Some quantifiers are used so frequently we use special symbols as a shorthand.

\[ \exists \] - “there exists”:

Let \( A \) be a set, and let \( p \) be a predicate such that for each \( x \in A \) the statement \( p(x) \) is a proposition.

\[ \exists x \in A, \ p(x) \] means “there exists an element \( x \) of \( A \) such that \( p(x) \) is true”.

**Examples:**

(i) \( \exists x \in \mathbb{R}, \ x^2 + x - 2 = 0 \) means “there exists a real number \( x \) such that \( x^2 + x - 2 = 0 \)” (this happens to be true, but that is not relevant here).

(ii) \( \exists x \in \mathbb{R}, \ x^2 < 0. \)
∀ - “for all”:

Let $A$ be a set, and let $p$ be a predicate such that for each $x \in A$ the statement $p(x)$ is a proposition.

∀ $x \in A$, $p(x)$ means “for all elements $x$ of $A$, $p(x)$ is true”.

Examples:

(i) ∀ $y \in \mathbb{R}$, $y^2 \geq 0$ means “every real number $y$ satisfies $y^2 \geq 0$”.

(ii) ∀ $n \in \mathbb{Z}$, $2n + 1$ is odd.

(iii) ∀ $x \in \mathbb{R}$, $x^2 > x$. 
Compound quantifiers

Statements often involve several quantifiers, e.g.,

\[ \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y > x. \]

Let \( A \) and \( B \) be sets, and let \( p \) be a predicate such that for each \( x \in A \) and \( y \in B \) the statement \( p(x, y) \) is a proposition.

(i) \( \forall x \in A, \exists y \in B, p(x, y) \) means “for all \( x \in A \) there exists \( y \in B \) such that \( p(x, y) \) is true”

(ii) \( \exists x \in A, \forall y \in B, p(x, y) \) means “there exists \( x \in A \) such that for all \( y \in B \), \( p(x, y) \) is true”

(iii) \( \forall x \in A, \forall y \in B, p(x, y) \) means “for all \( x \in A \) and \( y \in B \), \( p(x, y) \) is true”.

(iv) \( \forall x \in A, \forall y \in B, p(x, y) \) means “there exist \( x \in A \) and \( y \in B \) such that \( p(x, y) \) is true”.

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The order in which the quantifiers are written is very important. Consider:

(a) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y > x$

(b) $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}, y > x$

Proposition (a) is clearly true, whilst (b) is false.

*Examples on board*
1.1: Logical connectives and truth tables

We break statements down into simpler substatements using *logical connectives.*

Label statements by letters, e.g., $p$, $q$, $r$, etc.

Analyse the relationship between these substatements using *truth tables.* In a truth table, we consider assignments of ‘truth values’ $T$ (true) or $F$ (false) to each substatement and record the resulting truth value of the compound statement. A row corresponds to a choice of truth values for all the substatements.

If there are $n$ substatements, then there are $2^n$ possible assignments of truth values.
and

Consider the statement $r$: “$\pi$ lies between 3 and 4”. Then $r$ is “$p$ and $q$”, where $p$ is “$3 < \pi$” and $q$ is “$\pi < 4$”. We write $r = p \land q$.

We call $\land$ a logical connective, and it can be described by the following truth table, where $p$ and $q$ are statements. (The other fundamental logical connectives are $\lor$ and $\neg$, which we will describe shortly).

Let $p$ and $q$ be statements. “$p$ and $q$”, written $p \land q$” is given by the following truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Let $p$ and $q$ be statements. We describe “$p$ or $q$”, written $p \lor q$, by the following truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Note that this is different from the colloquial use of the word “or”, as it allows the possibility that both $p$ and $q$ are true.
Let $p$ be a statement. We denote the negation of $p$ by $\neg p$ (pronounced “not $p$”). It has the following truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

For example, if $p$ is “$\pi > 3$”, then $\neg p$ is $\pi \leq 3$. 
Examples

(i) Let $p$ and $q$ be statements. Find $\neg(p \land q)$.

On board

(ii) Let $p$ and $q$ be statements. Find $(\neg p) \lor (\neg q)$.

On board

Notice that $\neg(p \land q)$ and $(\neg p) \lor (\neg q)$ have the same truth table. Write $\neg(p \land q) \equiv (\neg p) \lor (\neg q)$.

Definition Statements are called equivalent if they have the same truth table. We use the symbol $\equiv$ to denote this.
More Examples

(iii)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$\neg (\neg p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

So $p \equiv \neg (\neg p)$ (they are equivalent).

(iv) Let $p$, $q$ and $r$ be statements. Find the truth table for $(p \land q) \lor r$.

On board