Towards the inclusion-exclusion principle

Theorem

Let $X$ and $Y$ be finite sets such that $X \cap Y = \emptyset$ (we say that $X$ and $Y$ are disjoint).
Then $|X \cup Y| = |X| + |Y|$.

Corollary

Suppose that $X_1, \ldots, X_n$ are pairwise disjoint finite sets.
Then $X_1 \cup \cdots \cup X_n = \bigcup_{i=1}^{n} X_i$ is a finite set and

$$|\bigcup_{i=1}^{n} X_i| = \sum_{i=1}^{n} |X_i|.$$
Theorem (Inclusion-exclusion principle)

Let $X$ and $Y$ be finite sets. Then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$ 

Proof.

$X \cup Y = (X \setminus Y) \cup (Y \setminus X) \cup (X \cap Y)$, a union of pairwise disjoint sets. 

By Corollary

$$|X \cup Y| = |X \setminus Y| + |Y \setminus X| + |X \cap Y|.$$ 

$X = (X \setminus Y) \cup (X \cap Y)$, so by Theorem

$$|X| = |X \setminus Y| + |X \cap Y|.$$ 

Similarly $|Y| = |Y \setminus X| + |X \cap Y|$. Substituting,

$$|X \cup Y| = (|X| - |X \cap Y|) + (|Y| - |X \cap Y|) + |X \cap Y| = |X| + |Y| - |X \cap Y|.$$
Theorem

Let \( X \) and \( Y \) be finite sets, with \( |X| = m \) and \( |Y| = n \). Then \( X \times Y \) is a finite set and

\[ |X \times Y| = mn. \]

Proof.

Suppose first \( X = \emptyset \) or \( Y = \emptyset \). Then \( X \times Y = \emptyset \), so \( |X \times Y| = 0 \).

Suppose now \( X \neq \emptyset \) and \( Y \neq \emptyset \).

Write \( X = \{x_1, \ldots, x_m\} \). Then

\[ X \times Y = \bigcup_{i=1}^{m} (\{x_i\} \times Y). \]

Now \( \forall \ i, \ |\{x_i\} \times Y| = |Y| \) (an easy exercise).

\( \{x_i\} \times Y \) are pairwise disjoint, so the result follows.
Corollary

Let $X_1, \ldots, X_m$ be finite sets, where $|X_i| = n_i$ for each $i$. Then

$$|X_1 \times \cdots \times X_m| = n_1 n_2 \cdots n_m.$$ 

Proof.

Use previous Theorem and induction on $m$. 

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We can now use these results to count functions.

**Corollary**

Let $X$ and $Y$ be non-empty finite sets, where $|X| = m$ and $|Y| = n$. Then the number of functions $X \rightarrow Y$ is $n^m$.

**Proof.**

Write $X = \{x_1, \ldots, x_m\}$.

A function $f : X \rightarrow Y$ determines an element

$$(f(x_1), f(x_2), \ldots, f(x_m)) \in Y^m = Y \times \cdots \times Y.$$

Conversely, each $(y_1, \ldots, y_m) \in Y^m$ determines a function $f : X \rightarrow Y$ by $\forall \ i, f(x_i) = y_i$.

So (given our labelling $x_1, \ldots, x_m$) we have constructed a bijection

$$g : \{f : X \rightarrow Y \text{ is a function}\} \rightarrow Y^m.$$
**Notation:** For \( n \in \mathbb{N} \), define \( n! = n(n - 1) \ldots 2 \cdot 1 \). Define \( 0! = 1 \)

**Theorem**

Let \( A \) and \( B \) be finite sets with \( |A| = |B| = n \). Then there are precisely \( n! \) bijections \( A \to B \).

In particular, there are precisely \( n! \) permutations of \( A \).

**Proof.**

Induction on \( n \). The result is clear for \( n = 1 \). Suppose that the result is true for \( n = k \).

Suppose \( |A| = |B| = k + 1 \). Fix \( a \in A \).

For each \( b \in B \), count bijections \( f : A \to B \) for which \( f(a) = b \).

By assumption, there are \( k! \) bijections \( A \setminus \{a\} \to B \setminus \{b\} \).

So there are \( k! \) bijections \( f : A \to B \) satisfying \( f(a) = b \).

There are \( k + 1 \) choices for \( b \), so the total number of bijections is \( (k + 1)k! = (k + 1)! \).

So by induction the result is true for all \( n \).