Let $A$ be a set and let $n \in \mathbb{N}$. We say $A$ has \textit{cardinality} $n$ if there exists a bijection $f : \mathbb{N}_n \rightarrow A$.

In this case write $|A| = n$.

Define $|\emptyset| = 0$.

\textbf{Example:}

Let $A = \{3, \pi, \sqrt{2}, 7\}$. Define $f : \mathbb{N}_4 \rightarrow A$ by

\begin{align*}
  f(1) &= 3 \\
  f(2) &= 7 \\
  f(3) &= \pi \\
  f(4) &= \sqrt{2}.
\end{align*}

$f$ is a bijection, so $|A| = 4$.

If $|A| = n$ for some $n \in \mathbb{N} \cup \{0\}$, then we say that $A$ is \textit{finite}. Otherwise we say $A$ is \textit{infinite}.
The definition of $|A|$ involved choice of $f : \mathbb{N}_n \to A$.

It’s not trivial to show that $|A|$ does not depend on the choice of $f$.

In proving this, we will derive results such as the pigeonhole principle.

**Theorem**

Let $m, n \in \mathbb{N}$. If there is a 1-1 function $f : \mathbb{N}_m \to \mathbb{N}_n$, then $m \leq n$.

**Proof.**

On board.
Corollary

Let $A$ be a set. Suppose that $m, n \in \mathbb{N}$ and that there are bijections $f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_n \to A$. Then $m = n$.

Proof.

Let $A$ be a set. Suppose that $m, n \in \mathbb{N}$ and that there are bijections $f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_n \to A$. Now $g^{-1} \circ f : \mathbb{N}_m \to \mathbb{N}_n$ is a bijection, and so 1-1. Hence by the Theorem $m \leq n$.

On the other hand, $f^{-1} \circ g : \mathbb{N}_n \to \mathbb{N}_m$ is also 1-1, so by the Theorem $n \leq m$. So $m = n$.

We have proved that the cardinality of a finite set is well-defined, i.e., it does not depend on choice.
Corollary

Let $A$ and $B$ be finite sets and let $f : A \to B$ be a 1-1 function. Then $|A| \leq |B|$.

Proof.

Write $|A| = m$ and $|B| = n$, so there are bijections $\alpha : \mathbb{N}_m \to A$ and $\beta : \mathbb{N}_n \to B$.

Then $\beta^{-1} \circ f \circ \alpha : \mathbb{N}_m \to \mathbb{N}_n$ is 1-1. Hence by the Theorem $m \leq n$.

Theorem (Pigeonhole principle)

Let $A$ and $B$ be non-empty finite sets and let $f : A \to B$. If $|A| > |B|$, then there exist $x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Proof.

The contrapositive of the Corollary above.

Example: In a group of 13 people, 2 will have a birthday in the same month.
Definition
Let $A$ be a set. Say $A$ is countable if it is finite or if there exists a bijection $f : \mathbb{N} \to A$. In the latter case we say that $A$ is countable infinite.

Remark: An example of an uncountable set is $\mathbb{R}$.

Example:
$\mathbb{Z}$ is countable. To see this, take the bijection $f : \mathbb{N} \to \mathbb{Z}$ defined as follows:

\[
\begin{align*}
  f(1) &= 0 \\
  f(2) &= 1 \\
  f(3) &= -1 \\
  f(4) &= 2 \\
  f(5) &= -2 \\
  f(6) &= 3 \\
  f(7) &= -3 \\
  f(8) &= 4 \\
  f(9) &= -4 \\
  etc.
\end{align*}
\]
Towards the inclusion-exclusion principle

**Theorem**

Let $X$ and $Y$ be finite sets such that $X \cap Y = \emptyset$ (we say that $X$ and $Y$ are disjoint). Then $|X \cup Y| = |X| + |Y|$.

**Proof.**

*Sketch proof on board, full proof in notes.*

**Corollary**

Suppose that $X_1, \ldots, X_n$ are pairwise disjoint finite sets. Then $X_1 \cup \cdots \cup X_n = \bigcup_{i=1}^{n} X_i$ is a finite set and $|\bigcup_{i=1}^{n} X_i| = \sum_{i=1}^{n} |X_i|$.

**Proof.**

By induction on $n$, using Theorem above.