6 - CARDINALITY OF SETS

6.1 Let $A = \{a, b, c, d\}$ and $B = \{e, f\}$. What is the cardinality of the Cartesian product $A \times B$? Write down an explicit bijection $\mathbb{N}_n \to A \times B$ where $n = |A \times B|$.

6.2 Suppose that $A$, $B$ and $C$ are finite sets. Prove that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$ 

[Hint: Apply a result from lectures to $A \cup B \cup C = (A \cup B) \cup C$, and then apply it again.]

6.3 In a class of 30 students, everyone supports at least one of three teams: 16 support $X$, 17 support $Y$ and 14 support $Z$; also 8 support both teams $X$ and $Y$, 7 support both $X$ and $Z$, and 9 support both $Y$ and $Z$. How many support all three teams?

6.4 Suppose $A$ is a set with $|A| = n$. How many functions are there form $A$ to $A$ which are not bijections?

6.5 Let $A$ and $B$ be finite sets. What is the number of 1-1 functions $A \to B$?

6.6 Suppose $n$ and $r$ are natural numbers with $n \geq r$. Prove that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$ 

[Hint: Let $A$ be a set with $|A| = n$. Then $\binom{n}{r}$ is the number of subsets of $A$ of cardinality $r$. Choose $a \in A$ and write $B = A \setminus \{a\}$. Count the subsets of $A$ of cardinality $r$ which are contained in $B$, and those which are not contained in $B$.]

6.7 (i) Calculate

(a) $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$ and

(b) $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$.

(ii) Expand $(1+x)^n$ using the binomial theorem. Hence (by making cunning choices for $x$) show that

(a) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ and

(b) $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$. 

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