Blocks with extraspecial defect groups of finite quasisimple groups [1]

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Abstract

We classify all blocks of finite quasisimple groups with extraspecial defect groups.

1 Introduction

In the modular representation theory of finite groups, a key theme is the determination of information about blocks from local information such as the defect groups and their normalizers. A number of important conjectures, such as those of Alperin, Dade and Broué, embody this theme. One approach to these conjectures is to use the classification of finite simple groups, and this is often done with some restriction on the defect groups. The class of extraspecial p-groups includes the smallest nonabelian p-groups and as such blocks with extraspecial defect groups are an obvious object of study. We determine the blocks of quasisimple groups with extraspecial defect groups, and it is hoped that this may prove useful in the application of the clasification of finite simple groups.

In work on nilpotent and controlled blocks of finite quasisimple groups in [6] and [4], we notice that blocks with extraspecial defect groups play a special role. In particular it turns out that almost all controlled blocks with nonabelian defect groups of a quasisimple group have extraspecial or trivial intersection defect groups. We apply some techniques developed in the study of nilpotent and controlled blocks of finite groups of Lie type to the classification of blocks of quasisimple groups whose defect groups are extraspecial.

In order to use inductive arguments, in treating the groups of Lie type we consider a broader class of p-groups: those whose derived subgroups have order 1 or p. We call such groups *small derived subgroup* groups, or *SDS* groups. The strategy is to prove strong results concerning classical groups (and certain related groups) in order to apply these results to the exceptional groups, whose blocks are not otherwise accessible for study. Our method for studying the blocks of the exceptional groups of Lie type is to consider the centralizer of an element in a defect group, which decomposes into classical groups and exceptional groups of smaller Lie rank.

Note that subgroups of SDS groups are themselves SDS groups, and if P is an SDS group and $Z \leq Z(P)$, then P/Z is also an SDS group. The principal examples we have in mind are extraspecial groups and abelian groups.

Let G be a finite group and p a prime. Although the classification concerns only blocks with respect to a field of characteristic p, we use methods from ordinary character theory, for example canonical characters, and so must use a p-modular system. Let $\mathcal{O}$ be a local discrete valuation ring, complete with respect to the p-adic valuation, with field of fractions $\mathcal{K}$ of characteristic zero and algebraically closed residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p. We assume that $\mathcal{O}$ contains a primitive $|G|$th root of

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unity. Write $\mathrm{Blk}(G)$ for the set of blocks of $\mathcal{O}G$ and denote by $B_0(G)$ the principal block of G.

Let N be a normal subgroup of G and write $\mathrm{Irr}(G)$ the set of irreducible $\mathcal{K}$-characters of G. For $\theta \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G \mid \theta)$ the subset of $\mathrm{Irr}(G)$ consisting of characters covering θ. We denote by $\mathrm{Irr}(B)$ the set of irreducible characters belonging to B, $k(B) = |\mathrm{Irr}(B)|$, and combine the above notations freely.

We use the convention $[x, y] = xyx^{-1}y^{-1}$.

In Section 2 we consider blocks of the symmetric and alternating groups and their covering groups. In Section 3 we list the extraspecial defect groups of the sporadic simple groups and their covers. In Section 4 we treat the classical groups. Here we prove stronger results than are necessary for the classification of their extraspecial defect groups, in order to provide the information necessary for the case of the exceptional groups of Lie type, which are considered in Section 5. Section 5 concludes with the exceptional covers of the alternating groups and groups of Lie type.

2 Symmetric and alternating groups

Let $G = \tilde{S}_n$, the double cover of S_n, where $n \geq 5$, and $\tilde{A}_n = K \leq G$. Let B be a p-block of K and B_G a block of G covering B. Suppose that B_G has noncentral defect group D_G, so that $D := D_G \cap K$ is a non-central defect group for B. Then D_G is isomorphic to a Sylow p-subgroup of $\tilde{S}_{pm}$, where m is the weight of B_G (see [13]). Hence $D_G Z(G)/Z(G) \cong D_1 \times \cdots \times D_t$, where $D_i \cong \mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p$, each D_i is nontrivial and at most $p-1$ of the D_i are isomorphic.

If p is odd, then $D_G Z(G)/Z(G) \cong D_G = D$. It is immediate that D is SDS if and only if $D_i \cong \mathbb{Z}_p$ for each i, i.e., if $t = m < p$. Hence all SDS defect groups are abelian when p is odd.

Suppose $p = 2$. Then $D_G Z(G)/Z(G)$ is SDS if and only if $m \leq 3$. Also, $DZ(G)/Z(G)$ is SDS if and only if $m \leq 3$. Note that since $[G : K] = 2$, B_G is the unique block of G covering B. The blocks of G (resp. K) are in 1-1 correspondence with the blocks of $G/Z(G)$ (resp. $K/Z(G)$).

Now B_G has an irreducible character χ labeled by a non-selfassociate partition, so by [23, 2.5.7] χ covers a G-stable irreducible character of B. Hence B is G-stable, and $[D_G : D] = 2$.

We treat the cases $m = 1$, $m = 2$ and $m = 3$ in turn.

Suppose $m = 1$. Then $D_G Z(G)/Z(G) \cong \mathbb{Z}_2$ and $|D_G| = 4$, so D_G is abelian.

Suppose $m = 2$. Then $D_G Z(G)/Z(G) \cong D_8$. This occurs precisely when $n = (r^2 + r + 8)/2$ for some r. In this case $DZ(G)/Z(G) \cong (\mathbb{Z}_2)^2$. Here $D_G Z(G)/Z(G)$, $DZ(G)/Z(G)$ and D are SDS, but not D_G.

Suppose $m = 3$. Then $D_G Z(G)/Z(G) \cong D_8 \times \mathbb{Z}_2$. This occurs precisely when $n = (r^2 + r + 12)/2$ for some r. Since $p = 2$ and m is odd, no non-faithful irreducible character of B_G is labeled by a selfassociate partition. Hence every non-faithful irreducible character of B_G covers a stable irreducible character of B, and so every non-faithful irreducible character of B is G-stable. Since there is an irreducible character of B_G of non-maximal defect, it follows that there is also a non-faithful irreducible character of B of non-maximal defect (recall that the defect of an irreducible character

θ of a finite group H is the non-negative integer d such that $|H|/\theta(1)_p = p^d$). Since Brauer's height zero conjecture holds for the alternating groups (see [28, 4.8]) it follows that $DZ(G)/Z(G)$ is nonabelian and so isomorphic to D_8. Here $D_G Z(G)/Z(G)$, $DZ(G)/Z(G)$ and D are SDS, but not D_G.

3 Sporadic groups

In Table 3 we list the blocks of quasisimple groups G such that $G/Z(G)$ is a sporadic simple group, with defect groups D such that $\overline{D} := DZ(G)/Z(G)$ is extraspecial. In the table, G is taken to be a full cover of the simple group. Except in the case $G/Z(G) \cong Co_1$, all of the information may be extracted either from the library in [20] or from other sources which are listed in the table.

$G/Z(G)$	$\overline{D}$ (faithful blocks)	$\overline{D}$ (nonfaithful blocks)	reference
M_{11}	none		[20]
M_{12}	3_+^{1+2}	3_+^{1+2}	[20]
M_{22}	none	none	[20]
M_{23}	none		[20]
M_{24}	3_+^{1+2}		[20]
J_1	none		[20]
J_2	3_+^{1+2}	3_+^{1+2}	[20]
J_3	none	none	[20]
J_4	3_+^{1+2}, 3_+^{1+2}, 11_+^{1+2}		[9]
HS	5_+^{1+2}	5_+^{1+2}	[20]
McL	5_+^{1+2}	5_+^{1+2}	[20]
Suz	D_8		[20]
Ly	none		[29]
He	D_8, 3_+^{1+2}, 7_+^{1+2}		[20]
Ru	3_+^{1+2}, 5_+^{1+2}	3_+^{1+2}, 5_+^{1+2}	[20]
$O'N$	7_+^{1+2}	D_8, 7_+^{1+2}	[20]
Co_3	5_+^{1+2}		[20]
Co_2	5_+^{1+2}		[20]
Co_1	D_8	3_+^{1+2}	[8] and see below
Fi_{22}	none	none	[20]
Fi_{23}	D_8		[7]
Fi_{24}'	D_8, 7_+^{1+2}	D_8, 7_+^{1+2}	[5]
Th	5_+^{1+2}		[30]
HN	none		[20]
$F_2 = B$	D_8	none	[10] and [25]
$F_1 = M$	3_+^{1+2}, 13_+^{1+2}		[11] and [25]

Table 1: Extraspecial defect groups of covering groups of sporadic groups

In [8] all of the blocks of Co_1 are given, as well as the 3-blocks for $2.Co_1$. Suppose $G \cong 2.Co_1$. It remains to check the case $p = 5$. By [8] there is precisely one conjugacy class of radical extraspecial 5-subgroups of $G/Z(G)$ and this is self-centralizing, with normalizer $5_+^{1+2} : GL_2(5)$. Let $Q \leq G$ such that $Q \cong 5_+^{1+2}$. Since $N_{G/Z(G)}(Z(G)Q/Z(G)) = N_G(Q)/Z(G)$ it follows that $N_G(Q)$ is a double cover of $5_+^{1+2} : GL_2(5)$ and $C_G(Q) = Z(G) \times Z(Q)$. Write $H = Z(G) \times Q \lhd N_G(Q)$. Since $C_G(Q) \leq H$, it follows that each of the two blocks of H are covered by a unique block of $N_G(Q)$. Since each of the blocks of H is $N_G(Q)$-stable, they must be covered by blocks of maximal defect. It follows that $N_G(Q)$ has no block with defect group Q, and so by Brauer's first main theorem G has no block with defect group Q. Hence the blocks with extraspecial defect groups of G are as given in the table.

Using [20], [8], [5] and [25] we verify that the only blocks with extraspecial defect groups for groups with $G/Z(G)$ sporadic simple and $1 \neq O_p(Z(G))$ occur for $G/Z(G) \cong M_{12}$, HS, J_2 and Ru for $p = 2$, in which case $|D| = 2^3$, and for $G/Z(G) \cong Sz$, $O'N$ and Fi'_{24} for $p = 3$, in which case $|D| = 3^3$.

We now turn to the case that G is a quasisimple group such that $G/Z(G)$ is a sporadic simple group and $O_p(G) \neq 1$, and B is a block of G with extraspecial defect group D. We assume again that G is the full cover of $G/Z(G)$. Using the same references as in Table 3 we observe the following: when $G/Z(G) \in \{M_{12}, J_2, Ru, HS\}$, we have $|Z(G)| = 2$ and there is a block D with defect group $D \cong Q_8$; when $G/Z(G) \cong M_{12}$, we have $|Z(G)| = 12$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; when $G/Z(G) \cong Suz$, we have $|Z(G)| = 6$ and there is a non-faithful block with defect group $D \cong 3_+^{1+2}$; when $G/Z(G) \cong O'N$, we have $|Z(G)| = 3$ and there is a block with defect group $D \cong 3_+^{1+2}$ and a block with defect group $D \cong 3_+^{1+4}$; these account for all such blocks.

4 Classical groups

Let V be a linear, unitary, non-degenerate orthogonal or symplectic space over field $\mathbb{F}_q$, where q is a power of r for some prime $r \neq p$.

If V is orthogonal, then there is a choice of equivalence classes of quadratic forms. Write $\eta(V)$ for the type of V as defined in [19], so $\eta(V) = +$ or $-$. Let $\eta(V) = +$ if V is linear and $\eta(V) = -$ if V is unitary.

If V is non-degenerate orthogonal or symplectic, then denote by $I(V)$ the group of isometries on V and let $I_0(V) = I(V) \cap \mathrm{SL}(V)$. Then $I(V) = I_0(V) = \mathrm{Sp}_{2n}(q)$ if V is symplectic, $I(V) = \langle -1_V \rangle \times I_0(V)$ with $I_0(V) = \mathrm{SO}_{2n+1}(q)$ if V is a $(2n+1)$-dimensional orthogonal space and $I(V) = O^\eta(V) = O^\eta_{2n}(q)$ and $I_0(V) = \mathrm{SO}^\eta_{2n}(q)$ if V is a $2n$-dimensional orthogonal space with $\eta(V) = \eta$, where $\eta = +$ or $-$.

If V is a non-degenerate orthogonal or symplectic space, then denote by $J_0(V)$ the conformal isometries of V with square determinant, and $D_0(V)$ the special Clifford group of an orthogonal space V (cf. [19]).

We will follow the notation of [3], [12] and [18]. Let $G = \mathrm{GL}^\eta(V)$ or $I(V)$, and let $\mathcal{F}_q = \mathcal{F}_q(G)$ (resp. $\mathcal{F}_q^{p'}$) be the set of polynomials serving as elementary divisors for all semisimple elements (resp. semisimple p'-elements) of G (cf. [3, p.6]). Let d_Γ and δ_Γ be the degree and the reduced degree of $\Gamma \in \mathcal{F}_q$, and let ϵ_Γ be the sign given by [3, p.6]), so that $d_\Gamma = \delta_\Gamma$ when $G = \mathrm{GL}^\eta(V)$, $\epsilon_\Gamma = +$ when $G = \mathrm{GL}(V)$ and $\epsilon_\Gamma = +$ or $-$ according as δ_Γ is odd or even when $G = U(V)$. For $\Gamma \in \mathcal{F}_q$, let e_Γ be the multiplicative order of $\epsilon_\Gamma q^{\delta_\Gamma}$ modulo p or 4 according as p is odd or even, and let $m_\Gamma(s)$ be the multiplicity of Γ in s. Thus $e_\Gamma \delta_\Gamma = ep^{\alpha_\Gamma}\delta'_\Gamma$ with $p \nmid \delta'_\Gamma$, where $e = e_{X-1}$.

Given a semisimple element $s \in G$, there is a unique orthogonal decomposition $V = \sum_{\Gamma \in \mathcal{F}_q} V_\Gamma(s)$, with $s = \prod_{\Gamma \in \mathcal{F}_q} s(\Gamma)$, where the $V_\Gamma(s)$ are nondegenerate subspaces of V and $s(\Gamma) \in \mathrm{GL}(V_\Gamma(s))$, $\mathrm{U}(V_\Gamma(s))$ or $I(V_\Gamma(s))$ (depending on G) has minimal polynomial Γ. This is called the primary decomposition of s. Write $m_\Gamma(s)$ for the multiplicity of Γ in $s(\Gamma)$. We have $C_G(s) = \prod_{\Gamma \in \mathcal{F}_q} C_\Gamma(s)$, where $C_\Gamma(s) = I(V_\Gamma(s))$ or $\mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma})$ as appropriate.

Suppose $G = \mathrm{GL}_n^\eta(q) = \mathrm{GL}^\eta(V)$, and let B be a p-block of G with a defect group D and label (s, κ). Then

$$V = V_0 \perp V_+, \quad D = D_0 \times D_+, \quad s = s_0 \times s_+, \tag{4.1}$$

where $V_0 = C_V(D)$, $V_+ = [D, V]$, $s_0 \in G_0 = \mathrm{GL}^\eta(V_0)$ and $s_+ \in G_+ := \mathrm{GL}^\eta(V_+)$. So if $p = 2$, then $V_0 = C_V(D) = 0$, $V_+ = V$, $D = D_+$ and $s = s_+$. We also denote $\mathrm{GL}^\eta(V)$ by $G(V)$ and $\mathrm{SL}^\eta(V)$ by $S(V)$.

For integers c, m and the prime p, we write $p^c \| m$ when $p^c \mid m$ and $p^{c+1} \nmid m$, and we let $p^a \| (q^2 - 1)$ or $2^{a+1} \| (q^2 - 1)$ according as p is odd or even.

Proposition 4.1 *Let $K := \mathrm{SL}^\eta(V) = \mathrm{SL}^\eta(n, q) \leq G := \mathrm{GL}^\eta(V)$, $Z \leq O_p(Z(K))$ and let $B_K \in \mathrm{Blk}(K)$ have defect group D_K. Let $B_G \in \mathrm{Blk}(G)$ be a weakly regular cover of B_K and D_G be a defect group of B_G, so that $D_K = D_G \cap K$. Suppose that D_K/Z is an SDS group. Then D_K is abelian if and only if D_G is abelian.*

(a) *If D_K/Z if nonabelian, then one of the following holds.*

  (i) *$p = 3$, $Z = 1$, $3 \| (q - \eta)$, $n = 3\delta_\Gamma$ with $3 \nmid \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^{p'}$, $D_G \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and $D_K/Z = D_K \cong 3_+^{1+2}$.*

  (ii) *$p = 2$, $n = 2\delta$ with odd $\delta = \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^{p'}$, $D_G = SD_{2^{a+2}}$ or $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ according as $2 \| (q - \eta)$ or $2^a \| (q - \eta)$ and $a = 2$ or 3. We have $(Z, a) = (1, 2)$ or $(\mathbb{Z}_2, 3)$. In the former case $D_K = D_K/Z \cong Q_8$ and in the latter case $D_K \cong Q_{2^{a+1}}$, generalized quaternion of order $2^{a+1} = 16$, and $D_K/Z \cong D_8$.*

(b) *If D_K/Z is abelian but D_G is nonabelian, then $Z = Z(D_K)$ and one of the following holds.*

  (i) *$p = 3$, $3 \| (q - \eta)$, $n = 3\delta_\Gamma$ with $3 \nmid \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^{p'}$, $D_G \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and $D_K \cong 3_+^{1+2}$.*

  (ii) *$p = 2$, $n = 2\delta$ with odd $\delta = \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^{p'}$, $D_G = SD_{2^4}$. In addition, $D_K \cong Q_8$.*

Conversely, suppose $p = 2$ or 3 and $3 \| (q - \eta)$ when $p = 3$. If $K = \mathrm{SL}^\eta(p\delta_\Gamma, q)$ for some $\Gamma \in \mathcal{F}_q^{p'}$ with $p \nmid \delta_\Gamma$, then there exists a block B_K satisfying either (a) or (b).

Proof: Note that if $Q \leq D_K$, then $Q/(Q\cap Z) \cong QZ/Z \leq D_K/Z$. So $Q/(Q\cap Z)$ is SDS.

Let (s,κ) be the label of $B := B_G$, and V, D, s have the corresponding decomposition (4.1). Let $s_+ = \prod_\Gamma s(\Gamma)$ be a primary decomposition, so that $V_+ = \bigoplus_\Gamma V_\Gamma$ with V_Γ the underlying space of $s(\Gamma)$. Thus

$$C_{G_+}(s_+) = \prod_\Gamma C_\Gamma, \quad C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma, q^{\delta_\Gamma}) \tag{4.2}$$

with $m_\Gamma = m_\Gamma(s_+)$. We may suppose $D_+ \in \mathrm{Syl}_p(C_{G_+}(s_+))$, that is, a Sylow subgroup of $C_{G_+}(s_+)$, so that

$$D_+ = \prod_\Gamma D_\Gamma, \quad D_\Gamma \in \mathrm{Syl}_p(C_\Gamma). \tag{4.3}$$

So D is a direct product of cyclic groups and wreath product p-groups. Denote by X_{p^α} a Sylow p-subgroup of the symmetric group S_{p^α}, that is, $X_{p^\alpha} \in \mathrm{Syl}_p(S_{p^\alpha})$. Here $X_{p^\alpha} = 1$ if $\alpha = 0$. Since $\mathrm{SL}^{\epsilon_\Gamma}(m_\Gamma, q^{\delta_\Gamma}) \leq C_\Gamma \cap S(V_\Gamma)$ and since $D_K = D_G \cap K$, it follows that D_K is abelian if and only if $m_\Gamma(s_+) < p$ for all Γ if and only if D_G is abelian.

Suppose D_K/Z is SDS and D_G is nonabelian. We have that

$$D_+ = D_1 \times D_2 \times \cdots \times D_m \tag{4.4}$$

with some D_i nonabelian, where $D_i = P_i \wr X_{p^{\alpha_i}}$, $P_i \cong \mathbb{Z}_{p^{c_i}}$ with $c_i \geq 1$ or $P_i \in \{\mathbb{Z}_{2^{c_i}}, SD_{2^{c_i+2}}\}$ with $c_i \geq 2$ according as $p \geq 3$ or $p = 2$. Without loss of generality take D_1 to be nonabelian. Let V_i be the underlying space of D_i, so that $X_{p^{\alpha_i}} \cong \mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p \leq S(V_i)$. Let

$$M(D_i) = \langle [D_i, D_i], X_{p^{\alpha_i}}, O_p(Z(S(V_i)))\rangle, \tag{4.5}$$

so that $M(D_i) \leq S(V_i)$ and hence $M(D_i) \leq D_K$. Since D_K/Z is SDS and since $[D_K, D_K]Z/Z \leq [D_K/Z, D_K/Z]$, it follows that $[D_K, D_K]Z/Z \leq \mathbb{Z}_p$ and in particular, $[D_K, D_K]$ is abelian of rank at most 2. Thus $[M(D_i), M(D_i)]Z_i/Z_i$ is cyclic of order 1 or p for some $Z_i \leq O_p(Z(S(V_i)))$.

Case 1. Suppose p is odd and $\alpha_i \geq 1$. Then $D_i = Y_i \rtimes X_{p^{\alpha_i}}$, where $Y_i = (\mathbb{Z}_{p^{c_i}})^{p^{\alpha_i}}$ is the base subgroup of D_i. Since $[X_{p^{\alpha_i}}, X_{p^{\alpha_i}}]Z_i/Z_i \cong [X_{p^{\alpha_i}}, X_{p^{\alpha_i}}]$ is cyclic, it follows that $\alpha_i = 1$ and $X_{p^{\alpha_i}} \cong \mathbb{Z}_p$. Take $\sigma \in X_{p^{\alpha_i}}$ such that σ acts on $Y_i = (\mathbb{Z}_{p^{c_i}})^p$ as the permutation $(12\ldots p)$.

Suppose first that $p \geq 5$. Let $(w, w^{-1}, 1, \ldots, 1) \in [D_i, D_i]$ with $|w| = p^{c_i}$, so that

$$[(w, w^{-1}, 1, \ldots, 1), \sigma] = (w^{-1}, w^2, w^{-1}, 1, \ldots, 1) \in [M(D_i), M(D_i)].$$

Similarly, $[(1, w, w^{-1}, 1, \ldots, 1), \sigma] = (1, w^{-1}, w^2, w^{-1}, 1, \ldots, 1) \in [M(D_i), M(D_i)]$. Since $[M(D_i), M(D_i)]Z_i/Z_i$ is cyclic of order 1 or p, it follows that

$$(w^{-1}, w^2, w^{-1}, 1, \ldots, 1)^t = z(1, w^{-1}, w^2, w^{-1}, 1, \ldots, 1)$$

for some $z \in Z_i \leq O_p(Z(S(V_i)))$, where $1 \leq t \leq (p-1)$, which is impossible.

Thus $p = 3$. Now $[(w, w^{-1}, 1), \sigma] = (w^{-1}, w^2, w^{-1}), [(1, w, w^{-1}), \sigma] = (w^{-1}, w^{-1}, w^2)$ and

$$[(w, 1, w^{-1}), \sigma] = (w^{-2}, w, w)$$

are elements of $[M(D_i), M(D_i)]$. But $[M(D_i), M(D_i)]Z_i/Z_i$ is cyclic of order 1 or p, so $c_i = 1$ and $D_i \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$. In particular, $D_1 \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$, and so $[D_1, D_1] = (\mathbb{Z}_3)^2$ and $D_1/Z(D_1) = 3^{1+2}_+$.

Suppose $3 \nmid (q - \eta)$, so that $O_3(Z(K)) = 1$ and D_K is SDS and nonabelian. In particular, $[D_K, D_K] \cong \mathbb{Z}_3$. If $D_i \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and $z_i \in Z(D_i) \setminus \{1_{V_i}\}$, then $C_{G(V_i)}(z_i) \cong \mathrm{GL}^{\epsilon}(3\delta, q^2)$ for some $\delta \geq 1$. Since $\mathrm{SL}(2, q)$ contains an element of order 3, it follows that we may suppose $w \in \mathrm{SL}(2, q)$ and hence $\det(w) = 1$. Thus $D_i \leq S(V_i)$, and $D_i \leq D_K$. But then $(\mathbb{Z}_3)^2 \cong [D_i, D_i] \leq [D_K, D_K] \cong \mathbb{Z}_3$, a contradiction. Thus $3 \mid (q - \eta)$. But then $3^a \mid |Z(D_i)|$, so $a = 1$ and $3 \| (q - \eta)$.

Suppose $m \geq 2$. Since D_2 is either cyclic or $D_2 \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$, it follows that there exists $x \in D_2$ such that $\det(x) = w^{-1}$ and $x^3 = 1$, where $w \in O_3(\mathbb{F}_q^\times)$ with $|w| = 3$. Thus $\det((w, 1, 1) \times x) = 1$, and

$$[(w, 1, 1) \times x, \sigma] = (w^{-1}, w, 1) \in [D_K, D_K],$$

where $(w, 1, 1) \in Y_1$ and $\sigma \in X_p \leq D_1$. Similarly, $[(1, w, 1) \times x, \sigma] = (1, w^{-1}, w) \in [D_K, D_K]$. Let

$$Q_1 = \langle (w, 1, 1) \times x, (1, w, 1) \times x, (1, 1, w) \times x, \sigma \rangle. \tag{4.6}$$

Then $Q_1 \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$, $Q_1 \leq D_1 \times \langle x \rangle$, so that $Q_1 \leq S(V_1 + V_2)$ and $[Q_1, Q_1] = [D_1, D_1] \leq [D_K, D_K]$.

If $m \geq 3$, then $Z \cap Q_1 = 1$. So $(\mathbb{Z}_3)^2 \cong [Q_1, Q_1]Z/Z = [Q_1Z/Z, Q_1Z/Z] \leq [D_K/Z, D_K/Z]$, contradicting our assumption that D_K/Z is SDS. Hence $m = 2$.

Suppose D_2 is nonabelian, so that $D_2 \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$. Take $x \in D_1$ with $\det(x) = w^{-1}$ and a similar proof to above shows that there exists a subgroup $Q_2 \leq \langle x \rangle \times D_2$ such that $Q_2 \leq S(V_1 + V_2)$ and $[Q_2, Q_2] = [D_2, D_2] \leq [D_K, D_K]$. This is impossible, since $[D_K, D_K]Z/Z \leq \mathbb{Z}_p$ and $[D_2, D_2] \cong [D_1, D_1] \cong (\mathbb{Z}_3)^2$. It follows that D_2 is cyclic of order $\mathbb{Z}_{3^c}$ with $c \geq a \geq 1$. Since $Z \leq O_3(Z(K)) \cong \mathbb{Z}_3$, it follows that $Z = 1$ or $O_3(Z(K))$. If $z \in Q_1 \cap Z$, then $z = \alpha 1_V$ and $z = z_1 \times z_2$, where $z_1 = \alpha 1_{V_1} \in Z(D_1)$ and $z_2 = \alpha 1_{V_2}$. Thus $\det(z_1) = \alpha^3 = 1$ and $\det(z_2) = 1$. Now $z_2 \in \langle x \rangle$ and $\det(x) = w^{-1}$, so $z_2 = x^i$ and $1 = \det(z_2) = \det(x)^i = w^{-i}$ and $3 \mid i$. Thus $z_2 = 1_{V_2}$ and $z = 1_V$. In particular, $Q_1 \cong Q_1Z/Z \leq D_K/Z$, which is impossible. Hence we cannot have $m \geq 2$ after all.

If $m = 1$, then $D_K = D_G \cap K = 3^{1+2}_+$ (since $D_G \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and D_K is a nonabelian SDS subgroup of D_G). Thus $Z = 1$ or $Z(D_K)$ according as D_K/Z is nonabelian or abelian. In the former case (a) (i) holds and in the latter (b) (i).

Case 2. Suppose $p = 2$. Then $D_i = Y_i \rtimes X_{2^{\alpha_i}}$, where $Y_i = (P_i)^{2^{\alpha_i}}$ is the base subgroup of D_i with $P_i \cong \mathbb{Z}_{2^{c_i}}$ or $SD_{2^{c_i+2}}, c_i \geq 2$. Since $[M(D_i), M(D_i)]$ is abelian and $X_{2^{\alpha_i}} \leq S(V_i)$, it follows that $\alpha_i \leq 2$.

Suppose $\alpha_i = 2$. Take $\sigma_j \in X_{2^2} \cong \mathbb{Z}_2 \wr Z_2 = D_8$ such that $\sigma_1 = (12), \sigma_2 = (34)$ and $\sigma_3 = (13)(24)$ acting on Y_i. Thus $(w, 1, w^{-1}, 1) \in [D_i, D_i]$ for any $w \in P_i$,

$$[(w, 1, w^{-1}, 1), \sigma_1] = (w^{-1}, w, 1, 1) \in [M(D_i), M(D_i)] \leq [D_K, D_K]$$

and $[(w,1,w^{-1},1),\sigma_2]=(1,1,w,w^{-1})\in[M(D_i),M(D_i)]\le[D_K,D_K]$. Thus

$$(w^{-1},w,1,1)=z(1,1,w,w^{-1})$$

for some $z\in Z(S(V_i))$. This is impossible, since we can choose $w\in P_i$ such that $|w|=4$. Thus $\alpha_i\le 1$ and so $D_i=P_i\wr\mathbb{Z}_2$ or P_i.

Suppose $P_i=SD_{2^{c_i+2}}\le G(U_i)$ and $\alpha_i=1$, where $U_i\perp U_i=V_i$. Then $P_i\cap S(U_i)$ contains a generalized quaternion group Q_i and for each $w\in Q_i$, $(w,w^{-1})\in[D_K,D_K]$. But $[D_K,D_K]$ is abelian, so this is a contradiction. Thus $\alpha_i=0$ and $D_i=SD_{2^{c_i+2}}=P_i$. So the generalized quaternion group $Q_{2^{c_i+1}}$ of order 2^{c_i+1} is a subgroup of $D_i\cap S(V_i)$ and so $Q_{2^{c_i+1}}\le D_K$. Since $D_i=SD_{2^{c_i+1}}$ is a Sylow 2-subgroup of $\mathrm{GL}^{\epsilon_\Gamma}(2,q^{\delta_\Gamma})$ for some Γ and $Z(G(V_i))\le\mathrm{GL}^{\epsilon_\Gamma}(2,q^{\delta_\Gamma})$, it follows that $2\|(q^{\delta_\Gamma}-\epsilon_\Gamma)$ and so $2\|(q-\eta)$. In particular, $O_2(Z(S(V_i)))\cong\mathbb{Z}_2$ and $[Q_{2^{c_i+1}},Q_{2^{c_i+1}}]\cong\mathbb{Z}_{2^{c_i-1}}\le[D_K,D_K]$. Now $[D_K,D_K]Z/Z\le\mathbb{Z}_2$ and $c_i\ge a\ge 2$, so $c_i-1=1$ or 2 and $c_i=2$ or 3.

Suppose $D_i\cong\mathbb{Z}_{2^{c_i}}\wr\mathbb{Z}_2\le G(V_i)$, so that $2^{c_i}\|(q^{\delta_\Gamma}-\epsilon_\Gamma)$ with $c_i\ge 2$. Suppose δ_Γ is odd, so that $c_i=a$. Since $(q-\eta)\mid(q^{\delta_\Gamma}-\epsilon_\Gamma)$ and δ_Γ is odd, it follows that $q^{\delta_\Gamma}\equiv\eta^{\delta_\Gamma}=\eta\equiv\epsilon_\Gamma\pmod{q-\eta}$, and $\eta-\epsilon_\Gamma=(q-\eta)t$ for some integer t. Thus $\eta=\epsilon_\Gamma$ except when $\eta=1,q=3$ and $\epsilon_\Gamma=-1$. But if $\eta=1$, then G is general linear and so $\epsilon_\Gamma=1$ for all Γ. Thus $\eta=\epsilon_\Gamma$, $2^a\|(q-\eta)$ and $Q_{2^{a+1}}=D_i\cap S(V_i)$ as $Z(G(V_i))\cong\mathbb{Z}_{q-\eta}$. But $O_2(Z(S(V_i)))\cong\mathbb{Z}_{\gcd(2\delta_\Gamma,q-\eta)}\cong\mathbb{Z}_2$, so $[Q_{2^{a+1}},Q_{2^{a+1}}]\cong\mathbb{Z}_2$ or $[Q_{2^{a+1}},Q_{2^{a+1}}]/\mathbb{Z}_2\cong\mathbb{Z}_2$. It follows that $a=2$ or 3 and $c_i=2$ or 3.

Suppose δ_Γ is even, so that $c_i=a+\alpha_\Gamma\ge a+1\ge 3$. Let $\delta_\Gamma^*=\delta_\Gamma/2$, so that $\mathrm{SL}(2,q^{\delta_\Gamma^*})\le S(U_i)$. Since $\mathrm{SL}(2,q^{\delta_\Gamma^*})$ contains an element w of order 2^{c_i}, it follows that $\det(w)=1$ and $D_i\cong\mathbb{Z}_{2^{c_i}}\wr\mathbb{Z}_2\le S(V_i)$. Thus $D_i\le D_K$ and $[D_i,D_i]\cong\mathbb{Z}_{2^{c_i}}\le[D_K,D_K]$. If $2\|(q-\eta)$, then $O_2(Z(S(V_i)))\cong\mathbb{Z}_2$ and $[D_i,D_i]/O_2(Z(S(V_i)))\cong\mathbb{Z}_{2^{c_i-1}}\not\cong\mathbb{Z}_2$. This is impossible, since $[D_K,D_K]Z/Z\le\mathbb{Z}_2$. If $2^a\|(q-\eta)$, then $[D_i,D_i]=\langle(w,w^{-1})\rangle\le Y_i$ with $|w|=2^{c_i}$ and $O_2(Z(S(V_i)))=\langle(w^{2^{c_i-a}},w^{2^{c_i-a}})\rangle\le Y_i$. Since $[D_K,D_K]Z/Z\le\mathbb{Z}_2$ and $[D_K,D_K]Z/Z\le Z(D_K/Z)$, it follows that

$$(w,w^{-1})^\sigma=(w^{-1},w)=z(w,w^{-1})$$

for some $z\in O_2(Z(S(V_i)))$, where $D_i=\langle Y_i,\sigma\rangle$ and σ acts on Y_i are the 2-cycle (12). If $z=(\alpha,\alpha)\in Y_i$, then $w^{-1}=\alpha w$ and $w=\alpha w^{-1}$, and so $w^{-2}=\alpha=w^2$. Thus $w^4=1$, which is impossible as $|w|=2^{c_i}\ge 8$. Thus δ_Γ is not even.

So if D_i is nonabelian, then $D_i=SD_{2^{a+2}}$ or $\mathbb{Z}_{2^a}\wr\mathbb{Z}_2$ according as $2\|(q-\eta)$ or $2^a\|(q-\eta)$, and in addition, $a=2$ or 3. Suppose $2^a\|(q-\eta)$ and $D_1\cong\mathbb{Z}_{2^a}\wr\mathbb{Z}_2$ and $D_2\cong\mathbb{Z}_{2^a}\wr\mathbb{Z}_2$. Then take $x_i=(w_i,1)\in Y_i$ with $|w_i|=2^a$ and $\sigma_i\in D_i$ acts as 2-cycle on the two factors of the base group Y_i. Then $\sigma_i\in S(V_i)$, $x_1\times x_2^{-1}\in S(V_1+V_2)$,

$$[x_1\times x_2^{-1},\sigma_1]=[x_1,\sigma_1]=(w_1,w_1^{-1})\in[D_K,D_K]\cap S(V_1)$$

and $[x_1^{-1}\times x_2,\sigma_2]=(w_2,w_2^{-1})\in[D_K,D_K]\cap S(V_2)$. Thus

$$H:=\langle(w_1,w_1^{-1})\rangle\times\langle(w_2,w_2^{-1})\rangle\le[D_K,D_K]. \tag{4.7}$$

Since $O_2(Z(S(V_1+V_2)))\le\mathbb{Z}_{2^a}$, it follows that $|H/O_2(Z(S(V_1+V_2)))|\ge 2^a$. But $[D_K,D_K]Z/Z\le\mathbb{Z}_2$, this is impossible. So each D_i is abelian when $i\ge 2$.

Suppose $2\|(q-\eta)$, and $D_1 = SD_{2^{a+2}}$ and $D_2 = SD_{2^{a+2}}$. The proof is similar to the case when $2^a\|(q-\eta)$. Take $x_i \in D_i$ such that $|x_i| = 2^{a+1}$ and $\det(x_1) = \det(x_2)$, and take $\sigma_i \in D_i \cap S(V_i)$ such that $x_i^{\sigma_i} = x_i^{2^a-1}$. Then

$$H := \langle [x_1 \times x_2, \sigma_1] \rangle \times \langle [x_1 \times x_2, \sigma_2] \rangle \leq [D_K, D_K]$$

and $|H| = 2^{2a}$. But $O_2(Z(S(V_1 + V_2))) \cong \mathbb{Z}_2$, so $H/O_2(Z(S(V_1 + V_2)))$ has order $2^{2a-1} \neq 2$. This is impossible, since $[D_K, D_K]Z/Z \leq \mathbb{Z}_2$. Thus each D_i is abelian when $i \geq 2$.

Suppose $m \geq 2$. A proof similar to that of Case 1 show there exists an element $x \in D_2$ and subgroup $Q \leq (D_1 \times \langle x \rangle) \cap S(V_1 \oplus V_2)$ such that $x^{2^c} = 1_{V_2}$, $\langle \det(x) \rangle = \Omega_{2^c}(D_2)$, and $Q \cong D_1$, where $2^c = |O_2(\mathbb{Z}_{q-\eta})|$. If $m \geq 3$, then $Q \cap Z = 1$, $Q = QZ/Z = Q \leq D_K/Z$, which is impossible. So $m = 2$.

If $2\|(q-\eta)$, then $D_1 = SD_{2^{a+2}}$. If $1 \neq z \in Q \cap Z$, then $z = z_1 \times z_2 := \alpha 1_{V_1} \times \alpha 1_{V_2}$ for some scalar α, where $z_i \in G(V_i)$. Since $z_i \in O_2(Z(G(V_i))) = \langle -1_{V_i} \rangle$ and $\dim V_1$ is even, it follows that $z_1 = -1_{V_1}$, $\det(z_1) = 1$, and so $\det(z_2) = 1$ and $z_2 = -1_{V_2}$. But $z_2 \in \langle x \rangle$, so $z_2 = x^i$, $1 = \det(z_2) = \det(x)^i$ and i is even. Thus $z_2 = x^i = 1_{V_2}$, and $Q = QZ/Z \leq D_K/Z$, which is impossible.

Hence $m = 1$. Then $D_G \cong \mathbb{Z}_{2^a} \wr Z_2$ or $SD_{2^{a+2}}$ according as $4\|(q-\eta)$ or $2\|(q-\eta)$. Thus $D_K = D_G \cap S(V) \cong Q_{2^{a+1}}$. If $a = 2$, then $D_K \cong Q_8$ and so $Z = 1$. If $a = 3$, then $Z \cong \mathbb{Z}_2$ and $D_K/Z \cong D_8$. Thus (a) (ii) holds.

Suppose $2^a\|(q-\eta)$, so that $D_1 \cong \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ and $|Z| \leq 2^a$. If $z \in Q \cap Z$ with $|z| = 2^k > 1$, then $z = z_1 \times z_2 := \alpha 1_{V_1} \times \alpha 1_{V_2}$ for some scalar α with $|\alpha| = 2^k$, where $z_i \in G(V_i)$. Then $1 = \det(z) = \det(z_1)\det(z_2)$ and $\det(z_2^{-1}) = \alpha^{2n_1}$, where $\dim V_1 = 2n_1$ with odd n_1. Since $z_2 \in \langle x \rangle$, it follows that $z_2 = x^i$ for some i. But then

$$\alpha^{2n_1} = \det(z_2)^{-1} = \det(x)^{-i} = w^{-i}$$

and $|w^i| = |\alpha^{2n_1}| = |\alpha^2| = 2^{k-1}$. Thus $|x^i| = |w^i| = 2^{k-1}$, $|z_2| = |x^i| = 2^{k-1}$ and so $|z_1| = |z_2| = |\alpha| = 2^{k-1}$. This is a contradiction, since $|z| = |\alpha| = 2^k$. It follows that $m = 1$ and (a), (b) hold.

Conversely, suppose $p = 2$ or 3 with $3\|(q-\eta)$ when $p = 3$. Let $K = \mathrm{SL}^\eta(p\delta_\Gamma, q) = \mathrm{SL}^\eta(V)$ for some $\Gamma \in \mathcal{F}_q^{p'}$ with $p \nmid \delta_\Gamma$ and $G = \mathrm{GL}^\eta(V)$. Take $s \in G$ such that $m_\Gamma(s) = p$ and so $C_G(s) = \mathrm{GL}^{\epsilon_\Gamma}(p, q^{\delta_\Gamma})$. Let $B_G = \mathcal{E}_p(G, (s))$ and B_K a block of K covered by B_G. Then $D(B_K)$ and $D(B_G)$ satisfy (a) and (b). $\square$

Let V be a non-degenerate orthogonal or symplectic space, $G = I_0(V)$ and let G^* be the dual group of G. Note that

$$\mathrm{Sp}_{2n}(q)^* = \mathrm{SO}_{2n+1}(q), \quad \mathrm{SO}_{2n+1}(q)^* = \mathrm{Sp}_{2n}(q), \quad \mathrm{SO}_{2n}^\eta(q)^* = \mathrm{SO}_{2n}^\eta(q).$$

If B is a block of $I_0(V)$, then there exists a semisimple p'-element $s \in I_0(V)^*$ such that

$$B \subseteq \mathcal{E}_p(I_0(V), (s)).$$

Let (D, b_D) be a Sylow B-subgroup of $I_0(V)$. Then V and D have corresponding decompositions

$$V = V_0 \perp V_+, \quad D = D_0 \times D_+. \tag{4.8}$$

If p is odd, then $V_0 = C_V(D)$, $V_+ = [\, V, D\,]$, $D_0 = \{1_{V_0}\}$ and $D_+ \leq I_0(V_+)$. If $p = 2$, then by [1, (5.1)], $D_0 \leq I_0(V_0)$ is an elementary abelian 2-subgroup and $D_+ \leq I_0(V_+)$. Let $G_0 := I_0(V_0)$, $G_+ := I_0(V_+)$, $C_+ := C_{I_0(V_+)}(D_+)$ and let V^* be the underlying space of $I_0(V)^*$.

Let $z \in D$ be a primitive element. If p is odd, then $z \in Z(D)$ with $|z| = p$ (cf. [19, p.176]). If $p = 2$, then z is given by the proof (3) of [1, Remark 2.2.9], so $|z| = 4$, $z \in K$ and $[\, V, D_+\,] = [\, V, z\,] = V_+$. Thus

$$z = z_0 \times z_+, \quad L := C_G(z) = L_0 \times L_+, \quad L_0 = G_0, \quad L_+ := \mathrm{GL}^\epsilon(m, q^e), \tag{4.9}$$

where $z_0 = 1_{V_0}$, $z_+ \leq D_+$ and $\dim V_+ = 2em$. Then L is a regular subgroup of G and we may suppose $s \in L^* \leq G^*$. In particular,

$$V^* = U_0 \perp U_+ \quad \text{and} \quad s = s_0 \times s_+, \tag{4.10}$$

where $U_0 = V_0^*$, $s_0 \in L_0^* = I_0(U_0)$, U_+ is the underlying space of L_+^* and $s_+ \in L_+^* \leq I_0(U_+)$.

Let $p = 2$ and let $s = \prod_\Gamma s(\Gamma)$ be the primary decomposition of s in $I_0(V^*)$, and let U_Γ be the underlying vector space of C_Γ. Then $C_{I_0(V^*)}(s) = \prod_\Gamma C_\Gamma$ and

$$C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma}) \quad \text{or} \quad I_0(U_\Gamma) \tag{4.11}$$

according as $\Gamma \neq X - 1$ or $\Gamma = X - 1$. In particular, $C_{I_0(V^*)}(s)$ is a regular subgroup of G^*.

Proposition 4.2 *Let $K := \Omega_{2n}^\eta(q) := \Omega^\eta(V) \leq G = \mathrm{SO}^\eta(V) \leq J := J_0(V)$, $Z \leq O_p(Z(K))$, $B_K \in \mathrm{Blk}(K)$, $B_G \in \mathrm{Blk}(G)$ covering B_K and $B_J \in \mathrm{Blk}(J)$ covering B_G. Let D_G, D_K and D_J be defect groups for B_G, B_K and B_J respectively. Then D_K is abelian if and only if D_G is abelian if and only if $D(B_J)$ is abelian.*

Suppose D_K/Z is SDS for some $Z \leq O_2(Z(K))$ and D_G is nonabelian. Then $p = 2$ and $(G, D_G, D_K, a, Z, D_K/Z)$ is listed in Table 2, where $\delta := \delta_\Gamma = \delta'$ or $2\delta'$ with odd δ' for some $\Gamma \in \mathcal{F}_q^{p'} \setminus \{X - 1\}$. In the last three cases, B_K is the principal block $B_0(K)$ of K.

Conversely, if $G = \mathrm{SO}^\eta(V)$ is given in Table 2 and $K = \Omega^\eta(V) \leq G$, then there exist blocks $B_K \in \mathrm{Blk}(K)$ and $B_G \in \mathrm{Blk}(G)$ covering B_K with defect groups D_K and D_G respectively as given in Table 2.

Proof: Let $B = B_G$ and $D = D_G$, so that we may suppose $D_K = D \cap K$ and $D = D_J \cap G$. Since G is self-dual, we have $V = V^*, U_0 = V_0, U_+ = V_+$ in (4.10) and $U_\Gamma = V_\Gamma$ in (4.11).

Write $K_0 = \Omega(V_0)$, $K_+ = \Omega(V_+)$, $K_\Gamma = \Omega(V_\Gamma)$ and $M_+ := \mathrm{SL}^\epsilon(m, q^e) \leq L_+ \cap K_+$, so that

$$K_0 \times M_+ \leq C_K(z) \leq L_0 \times L_+, \quad C_K(z) = \langle K_0 \times C_{K_+}(z_+), t_0 \times t_+ \rangle$$

and $[L_+ {:} C_{K_+}(z_+)] \leq 2$, where $t_0 \in L_0 \setminus K_0$ and $t_+ \in L_+$.

Case 1. Suppose p is odd. Since $|G{:}K| = 2$, it follows that $D = D_K$. Let (z, B_z) be a major subsection of B_K. Then B_z covers a block $B_0 \times B_+$ of $K_0 \times C_{K_+}(z_+)$ with

G	D_G	a	D_K	Z	D_K/Z
$\mathrm{SO}^+(4\delta', q)$	$\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$	2	$Q_8 \circ \mathbb{Z}_4$	1	$Q_8 \circ \mathbb{Z}_4$
$\mathrm{SO}^+(4\delta', q)$	$\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$	2	$Q_8 \circ \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{SO}^+(4\delta', q)$	$\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$	3	$Q_{2^{a+1}} \circ \mathbb{Z}_8$	$\mathbb{Z}_2$	$D_8 \times \mathbb{Z}_4$
$\mathrm{SO}^+(4\delta', q)$	$SD_{2^{a+2}}$	2	Q_8	1	Q_8
$\mathrm{SO}^+(4\delta', q)$	$SD_{2^{a+2}}$	2	Q_8	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{SO}^+(4\delta', q)$	$SD_{2^{a+2}}$	3	$Q_{2^{a+1}}$	$\mathbb{Z}_2$	D_8
$\mathrm{SO}^+(8\delta', q)$	$\mathbb{Z}_{2^{a+1}} \wr \mathbb{Z}_2$	2	$Q_{2^{a+2}} \circ \mathbb{Z}_8$	$\mathbb{Z}_2$	$D_8 \times \mathbb{Z}_4$
$\mathrm{SO}^+(8\delta', q)$	$SD_{2^{a+2}}$	2	$Q_{2^{a+2}}$	$\mathbb{Z}_2$	D_8
$\mathrm{SO}^-(4, q)$	$D_{2^{a+1}}.2$	2	D_8	1	D_8
$\mathrm{SO}^+(4, q)$	$2_+^{1+4}.2$	2	2_+^{1+4}	1	2_+^{1+4}
$\mathrm{SO}^+(4, q)$	$2_+^{1+4}.2$	2	2_+^{1+4}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Table 2: Defect 2-groups of $\Omega_{2n}^{\eta}(q)$ with quotient an SDS group

$B_0 \in \mathrm{Blk}(K_0)$ and $B_+ \in \mathrm{Blk}(C_{K_+}(z_+))$ such that D is a defect group for both B_z and $B_0 \times B_+$.

By [16, Lemma 4.1], there exists a B-subgroup (z, B_L) such that B_L covers B_z. Thus (z, B_L) is a major subsection of B. Since $L = L_0 \times L_+$ with $L_0 = G_0 = \mathrm{SO}(V_0)$, it follows that $B_L = B_{L_0} \times B_{L_+}$ with $B_{L_0} \in \mathrm{Blk}(G_0)$ and $B_{L_+} \in \mathrm{Blk}(L_+)$. But B_L covers B_z and B_z covers $B_0 \times B_+$, so B_{L_0} covers B_0 and B_{L_+} covers B_+. In particular, D_+ is a defect group for B_+ and B_{L_+}. Thus $D_+ \in \mathrm{Syl}_p(C_{L_+}(s_+))$. But $Z(K) = 1$ or $\mathbb{Z}_2$ and p is odd, so $D_+Z/Z \cong D_+$ and by Proposition 4.1, D_+ cannot be nonabelian and SDS.

By [19, (1A)], $C_J(z) = \langle L, \tau\rangle$, where $\tau = \tau_0 \times \tau_+$ with $[\tau_+, L_+] = 1$ and $J = \langle G, \tau\rangle$. Since $D_K = D = \langle 1_{V_0}\rangle \times D_+$ with $D_+ \leq L_+$, it follows that $[\tau, D] = 1$. But $D_J/D \leq \langle\tau\rangle$, it follows that D_K is abelian if and only if D_J is abelian, if and only if D is abelian.

Since $D \cong D_+$, it follows that if p is odd, then we cannot have a situation where D_K/Z is SDS and D_G is nonabelian.

Case 2. Suppose $p = 2$, so that $B = \mathcal{E}_2(G, (s))$ and $D \in \mathrm{Syl}_2(C_G(s))$. Follow the notation of (4.11), and let $B_\Gamma = \mathcal{E}_2(C_\Gamma, (s(\Gamma)))$ and $D_\Gamma := D(B_\Gamma)$. Thus $D = \prod_\Gamma D_\Gamma$, $D_\Gamma \in \mathrm{Syl}_2(C_\Gamma)$ and $D_K = D \cap K$.

Note that if D_K is abelian, then $m_\Gamma(s) \leq 1$ for each $\Gamma \neq X - 1$ and so D_Γ is abelian for $\Gamma \neq X - 1$. Hence in order to show that D_K is abelian if and only if D is abelian, it suffices to consider $\Gamma = X - 1$.

Write $2n_{X-1} = \dim V_{X-1}$ and $\eta_{X-1} = \eta(V_{X-1})$. Thus $C_{X-1} = \mathrm{SO}(V_{X-1})$. Let $Q_{X-1} \in \mathrm{Syl}_2(I(V_{X-1}))$, so that $D_{X-1} = Q_{X-1} \cap I_0(V_{X-1})$ and

$$Q_{X-1} = Q_0 \times Q_1 \times \cdots \times Q_m, \tag{4.12}$$

where $Q_0 = 1$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ according as $q^{n_{X-1}} \equiv \eta_{X-1} \pmod 4$ or $-\eta_{X-1} \pmod 4$, and $Q_i = D_{2^{a+1}} \wr X_{2^{\alpha_i}}$ with $X_{2^{\alpha_i}} \in \mathrm{Syl}_2(S_{2^{\alpha_i}})$. In addition, $Q_0 \cap I_0(V(0)) = \langle -1_{V(0)}\rangle$ and $-1_{V(0)} \notin \Omega(V(0))$ when $Q_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus if D_K is abelian, then $m_{X-1}(s) \leq 2$ or 4 according as according as $q^{n_{X-1}} \equiv \eta_{X-1} \pmod 4$ or $-\eta_{X-1} \pmod 4$. So D_K is abelian if and only if D is abelian.

It follows by [19, (1A)] that $C_J(s) = \langle C_G(s), \tau \rangle$ with $\tau \in J \setminus K$ and $[\tau, C_G(s)] = 1$. Thus $D_J \in \mathrm{Syl}_2(C_J(s))$ and D_J is abelian. Conversely, if D_J is abelian, then D and D_K are abelian, since $D = D_J \cap G$ and $D_K = D_J \cap K$. It follows that D_J is abelian if and only if D is abelian if and only if D_K is abelian, proving the first part of the proposition.

Now suppose that D_K is nonabelian and that D_K/Z is SDS for some $Z \leq Z(K)$. Then there exists $\Gamma \in \mathcal{F}$ such that $m_\Gamma(s) \geq 2$ or 4 according as $\Gamma \neq X-1$ or $\Gamma = X-1$. Since $Z = 1$ or $\mathbb{Z}_2$, it follows that $Z \leq D_K$. If $K \neq K_\Gamma$, then $Z \cap K_\Gamma = 1$, $D_\Gamma \cap K_\Gamma \cong (D_\Gamma \cap K_\Gamma)Z/Z \leq D_K/Z$ and in particular $D_\Gamma \cap K_\Gamma$ is SDS. If $K = K_\Gamma$, then $D_K/Z = (D_\Gamma \cap K_\Gamma)/Z$ and by assumption, this is SDS. Thus $(D_\Gamma \cap K_\Gamma)/(Z \cap K_\Gamma)$ is SDS in any case.

We treat the cases $m_\Gamma(s) \geq 2$ for some $\Gamma \neq X-1$, and $m_{X-1}(s) \geq 4$ separately.

Case 2.1 Suppose there is $\Gamma \neq X-1$ with $m_\Gamma(s) \geq 2$. Let

$$M_\Gamma = \mathrm{SL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma}) \leq C_{K_\Gamma}(s_\Gamma) \leq C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma}),$$

so that $Z \cap K_\Gamma \leq M_\Gamma$. Since $(D_\Gamma \cap M_\Gamma)/(Z \cap K_\Gamma) \leq (D_\Gamma \cap K_\Gamma)/(Z \cap K_\Gamma)$, it follows that $(D_\Gamma \cap M_\Gamma)/(Z \cap K_\Gamma)$ is SDS. Hence by Proposition 4.1 $D_\Gamma \cong \mathbb{Z}_{2^c} \wr \mathbb{Z}_2$ or $SD_{2^{c+2}}$, where $c = 2$ or 3 and $2^{c+1} \| (q^{2\delta_\Gamma} - 1)$. In addition, $[D_\Gamma{:}D_\Gamma \cap K_\Gamma] = 2$, since $[C_\Gamma{:}C_{K_\Gamma}(s_\Gamma)] = 2$. In particular, $m_\Gamma(s) = 2$ and $\eta(V_\Gamma) = (\epsilon_\Gamma)^{m_\Gamma(s)} = +1$. But $\dim(V_\Gamma) = 4\delta_\Gamma$, so $(-1)^{\frac{\dim(V_\Gamma)(q-1)}{4}} = +1 = \eta(V_\Gamma)$ and by [24, Proposition 2.5.13 (ii)], $-1_{V_\Gamma} \in K_\Gamma$.

Now consider $\Delta \in \mathcal{F}$ with $m_\Delta(s) \leq 1$. If $D_\Delta \neq 1$, then $D_\Delta = \langle -1_{V_\Delta} \rangle \in \mathrm{Syl}_2(C_\Delta)$ or $D_\Delta \cong \mathbb{Z}_{2^c}$ for some $c \geq 2$. In each case we will find an element $u_\Delta \in D_\Delta \setminus K_\Delta$.

Suppose $D_\Delta = \langle -1_{V_\Delta} \rangle \in \mathrm{Syl}_2(C_\Delta)$. In this case $C_\Delta = \mathrm{GL}^{\epsilon_\Delta}(1, q^{\delta_\Delta})$, $\eta(V_\Delta) = \epsilon_\Delta$ and $2 \| (q^{\delta_\Delta} - \epsilon_\Delta)$. If $\epsilon_\Delta = +1$, then $2 \| (q^{\delta_\Delta} - 1)$ and so δ_Δ is odd, $2 \| (q-1)$ as $(q-1) \mid (q^{\delta_\Delta} - 1)$ and $(q+1) \nmid (q^{\delta_\Delta} - 1)$. Thus $(-1)^{(2\delta_\Delta(q-1))/4} = -1 \neq \eta(V_\Delta)$ and $D_\Delta \cap K_\Delta = 1$. Similarly, if $\epsilon_\Delta = -1$, then $\eta(V_\Delta) = -1$, $2 \| (q^{\delta_\Delta} + 1)$ and so $4 \mid (q^{\delta_\Delta} - 1)$. Now $(-1)^{(2\delta_\Delta(q-1))/4} = -1$ if and only if δ_Δ is odd and $2 \| (q-1)$, which implies that $2 \| (q^{\delta_\Delta} - 1)$. So $(-1)^{(2\delta_\Delta(q-1))/4} = +1 \neq \eta(V_\Delta)$ and $D_\Delta \cap K_\Delta = 1$. It follows that

$$-1_{V_\Delta} \in D_\Delta \setminus K_\Delta$$

and we set $u_\Delta = -1_{V_\Delta}$.

If $D_\Delta \cong \mathbb{Z}_{2^c}$ for some $c \geq 2$, then $C_\Delta = \mathrm{GL}^{\epsilon_\Delta}(1, q^{\delta_\Delta})$ and $2^c \| (q^{\delta_\Delta} - \epsilon_\Delta)$. In this case, $-1_{V_\Delta} \in K_\Delta$ and there exists $u_\Delta \in D_\Delta \setminus K_\Delta$.

Now suppose that in addition we have $m_{X-1}(s) \geq 4$. Follow the notation of (4.12). Let $V(i)$ be the underlying space of Q_i and $V(+) = \oplus_{i \geq 1} V(i)$, so that $V_{X-1} = V(0) \perp V(+)$. If z_{X-1} is a primary element of Q_{X-1} as given in [1, Remark 2.2.9 (2)], then $|z_{X-1}| = 4$, $z_{X-1} \in I_0(V_{X-1})$, $z_{X-1} \in D_{X-1}$ and

$$z_{X-1} = 1_{V(0)} \times z_+(X-1) \quad C_{I_0(V_{X-1})}(z_{X-1}) = I_0(V(0)) \times L(+), \quad L(+) = \mathrm{GL}^{\epsilon}(w_{X-1}, q),$$

where $2w_{X-1} = \dim(V_{X-1}) - \dim(V(0))$ and $z_+(X-1) \in L(+) \leq I_0(V(+))$. If $a = 2$, then $z_{X-1} \notin K_{X-1}$. If $a \geq 3$, then $z_{X-1} \in K_{X-1}$ as z_{X-1} is a square of some 2-element of $I_0(V_{X-1})$ and there exists $u_{X-1} \in D_{X-1} \setminus K_{X-1}$.

Recall that $D_\Gamma \cong \mathbb{Z}_{2^c} \wr \mathbb{Z}_2$ or $SD_{2^{c+2}}$, where $c = 2$ or 3 and $2^c \| (q^{\delta_\Gamma} - \epsilon_\Gamma)$. In each case we define an element u_Γ of $D_\Gamma \setminus K_\Gamma$ as follows.

If $D_\Gamma \cong \mathbb{Z}_{2^c} \wr \mathbb{Z}_2 \le C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(2, q^{\delta_\Gamma})$, then take $u_\Gamma = \mathrm{diag}\{w, 1\} \in C_\Gamma$ such that w is an element of order 2^c in $\mathbb{F}^\times_{q^{\delta_\Gamma}}$, so that $-1_{V_\Gamma} \notin \langle u_\Gamma \rangle$ and $u_\Gamma \in D_\Gamma \setminus K_\Gamma$. If $D_\Gamma = SD_{2^{c+2}} \le C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(2, q^{\delta_\Gamma})$, then $SD_{2^{c+2}} = \langle x, y | x^{2^{c+1}} = y^4 = 1, y^{-1}xy = x^{2^c}x^{-1} \rangle$ with $\det(y) = 1$. Thus we take $u_\Gamma = x$ and so $u_\Gamma \in D_\Gamma \setminus K_\Gamma$.

Suppose $D_\Delta \ne 1$ for some $\Delta \ne \Gamma$. If $\Delta \ne X - 1$ and $m_\Delta(s) \ge 2$, then define $u_\Delta \in D_\Delta \setminus K_\Delta$ as for Γ. Otherwise, define u_Δ as above. Define

$$P := \langle D_\Gamma \cap K_\Gamma, u_\Gamma \times u_\Delta \rangle.$$

Then $P \le (D_\Gamma \times \langle u_\Delta \rangle) \cap \Omega(V_\Gamma + V_\Delta) \le D_K$. Since $[P, P] = [D_\Gamma, D_\Gamma] \le K_\Gamma$ and since $K_\Gamma \cap Z = 1$, it follows that $[D_\Gamma, D_\Gamma] \cong [P, P]Z/Z \le [D_K/Z, D_K/Z]$. This is impossible, since $[D_K/Z, D_K/Z] \le \mathbb{Z}_2$ and $[D_\Gamma, D_\Gamma] \cong \mathbb{Z}_{2^c}$. Thus $D_\Delta = 1$ for all $\Delta \ne \Gamma$.

Case 2.2 Suppose $D = D_\Gamma \in \{\mathbb{Z}_{2^c} \wr \mathbb{Z}_2, SD_{2^{c+2}}\}$ with $3 \ge c \ge a \ge 2$ and let $\delta = \delta_\Gamma$, so that $C = C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(2, q^\delta) \ge M_\Gamma = \mathrm{SL}(2, q^\delta)$. Since $K = [G, G]$, it follows that

$$Q_{2^{c+1}} = D \cap M_\Gamma \le D \cap K = D_K.$$

But C/M_Γ is cyclic and $D/Q_{2^{c+1}} \le C/M_\Gamma$, so D_K is the unique subgroup of D such that $Q_{2^{c+1}} \le D_K$ and $[D{:}D_K] = 2$. Since $[C{:}M_\Gamma Z(C)] = 2$, it follows that $D_K \in \mathrm{Syl}_2(M_\Gamma Z(C))$, $D_K \cong Q_{2^{c+1}} \circ \mathbb{Z}_{2^c}$ or $Q_{2^{c+1}}$ according as $D \cong \mathbb{Z}_{2^c} \wr \mathbb{Z}_2$ or $SD_{2^{c+2}}$.

If $\delta = \delta'$ is odd, then $c = a$ and $Z = 1$ or $Z(K)$. This implies the first 6 cases in Table 2. Suppose δ is even, so that $\alpha := \alpha_\Gamma \ge 1$ and $c = a + \alpha \ge a + 1 \ge 3$. Thus $c = 3$, $\alpha = 1$ and $\delta = 2\delta'$ for some odd δ'. In particular, $8 = 2^c \| q^\delta - 1$. Thus $(D, D_K) = (\mathbb{Z}_{2^c} \wr \mathbb{Z}_2, Q_{2^{c+1}} \circ \mathbb{Z}_{2^c})$ or $(SD_{2^{c+2}}, Q_{2^{c+1}})$ according as $\epsilon_\Gamma = +$ or $-$. In this case $Z = Z(G) \cong \mathbb{Z}_2$ and the cases 7 and 8 in Table 2 hold.

Case 2.3 Suppose that $m_{X-1}(s) \ge 4$ and that D_Γ is abelian for all $\Gamma \ne X - 1$ and D_{X-1} is nonabelian. Recall that $(D_{X-1} \cap K_{X-1})/(Z \cap K_{X-1})$ is SDS and $D_{X-1} \cap K_{X-1}$ is a Sylow 2-subgroup of K_{X-1}. Note that $w_{X-1} \ge 2$.

Let $M(+) = \mathrm{SL}^\epsilon(w_{X-1}, q) \le L(+) = \mathrm{GL}^\epsilon(w_{X-1}, q)$, so that $M(+) \le \Omega(V(+))$. Since $Z \cap \Omega(V(+)) \le Z(M(+))$, it follows that $D_{X-1} \cap M(+)/Z(M(+))$ is SDS. By Proposition 4.1 $D_{X-1} \cap L(+) = SD_{2^{a+2}}$ or $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ according as $2 \| (q - \epsilon)$ or $2^a \| (q - \epsilon)$. Since $4 \mid (q - \epsilon)$, it follows $D_{X-1} \cap L(+) \cong \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ with $a = 2$ or 3. In particular, $w_{X-1} = 2$.

A similar proof to that of Case 2.1 shows that $D_\Gamma = 1$ for any $\Gamma \ne X - 1$ and $D_0 = 1$. So $\dim V = 4 = m_{X-1}(s)$ and $K_{X-1} = K = \Omega_4^\eta(q)$. If $\eta = -$, then $K = K_{X-1} = \Omega_4^-(q) = P\mathrm{SL}_2(q^2)$, so that $D_K \cong Q_{2^{a+2}}/Z(\mathrm{SL}_2(q^2)) = D_{2^{a+1}}$ and $Z = 1$. Thus $a = 2$ and $D_K/Z = D_K = D_{2^3}$ and $D_G = D_8.2$.

If $\eta = +$, then $K = K_{X-1} = \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and $D_K \cong Q_{2^{a+1}} \circ Q_{2^{a+1}}$. If $a \ge 3$, then $D_K/Z(K)$ is nonabelian and not SDS. Thus $a = 2$, $D_K = 2_+^{1+4}$, $D_G = 2_+^{1+4}.2$, and $Z = 1$ or $Z(K) \cong \mathbb{Z}_2$.

Conversely, suppose $G = \mathrm{SO}^\eta(V)$ is as given in Table 2 and $K = \Omega(V) \le G$. If $\dim V = 4$, then let $B = B_0(K)$, $B_G = B_0(G)$ and so the defect groups D_G of B_G and D_K of B_K are as given in Table 2. If $\dim V > 4$, then take a semisimple $2'$-element $s \in G$ such that $m_\Gamma(s) = 2$ and let $B_G = \mathcal{E}_2(G, (s))$. Thus $C = C_G(s) = \mathrm{GL}^{\epsilon_\Gamma}(2, q^\delta)$

and $D_G \in \mathrm{Syl}_2(C)$ for some $D_G = D(B_G)$. If B_K is the block of K covered by B_G, then we may suppose $D_K = D_G \cap K$ for some $D_K = D(B_K)$. By Case 2.2, the defect groups D_G and D_K are as given in Table 2. □

Proposition 4.3 *Let $K := \Omega_{2n+1}(q) := \Omega(V)$ or $\mathrm{Sp}_{2n}(q) = \mathrm{Sp}(V)$, $G = \mathrm{SO}_{2n+1}(q) = \mathrm{SO}(V)$ or $\mathrm{Sp}_{2n}(q)$, and $J = \mathrm{SO}(V)$ or $J_0(V)$, so that $K \le G \le J$. Let $B_K \in \mathrm{Blk}(K)$, $B_G \in \mathrm{Blk}(G)$ covering B_K and $B_J \in \mathrm{Blk}(J)$ covering B_G. Let D_K, D_G and D_J be blocks of B_K, B_G and B_J respectively. Then D_K is abelian if and only if D_G is abelian if and only if D_J is abelian.*

Suppose D_G is nonabelian and D_K/Z is SDS for some $Z \le O_p(Z(K))$. Then $p = 2$ and $(G, D_G, D_K, a, Z, D_K/Z)$ is as listed in Table 3, where $D_K = D_G \cap K$ and $\delta = \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^{p'} \setminus \{X - 1\}$ with odd δ.

Conversely, if $G = \mathrm{SO}_{2n+1}(q)$ or $\mathrm{Sp}_2(q)$ is as given in Table 3 and $K = \Omega_{2n+1}(q)$ or $\mathrm{Sp}_2(q)$ such that $K \le G$, then there exist blocks $B_K \in \mathrm{Blk}(K)$ and $B_G \in \mathrm{Blk}(G)$ covering B_K with defect groups D_K and D_G as given in Table 3.

G	D_G	a	D_K	Z	D_K/Z
$\mathrm{Sp}(2,q)$	$Q_{2^{a+1}}$	2	$Q_{2^{a+1}}$	1	Q_8
$\mathrm{Sp}(2,q)$	$Q_{2^{a+1}}$	2	$Q_{2^{a+1}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{Sp}(2,q)$	$Q_{2^{a+1}}$	3	$Q_{2^{a+1}}$	$\mathbb{Z}_2$	D_8
$\mathrm{SO}(3,q)$	$D_{2^{a+1}}$	3	D_8	1	D_8
$\mathrm{SO}(4\delta+1,q)$	$\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$	2	$Q_{2^{a+1}} \circ \mathbb{Z}_4$	1	$Q_8 \circ \mathbb{Z}_4$
$\mathrm{SO}(4\delta+1,q)$	$SD_{2^{a+2}}$	2	$Q_{2^{a+1}}$	1	Q_8

Table 3: Defect 2-groups of $\mathrm{SO}(2n+1,q)$ and $\mathrm{Sp}(2n,q)$ with quotient a SDS group

PROOF: For p odd, the proof is similar to the proof of Case 1 of Proposition 4.2 and for reasons of space we omit it.

Suppose $p = 2$, so that $B_G = \mathcal{E}_2(G,(s))$. Let s^* be a dual element of s in $I_0(V)$ given by [3, (4A)]. Then $D_G \in \mathrm{Syl}_2(C_G(s^*))$. A proof similar to the Case 2 of proof of Proposition 4.2 with s replaced by s^* shows that Proposition 4.3 holds. Note that $\Omega_3(q) \cong P\mathrm{SL}_2(q)$ and $Z(\Omega_{2n+1}(V)) = 1$. □

Proposition 4.4 *Let $K := \mathrm{Spin}^\eta(V) \lhd J$ such that J/K is abelian, $C_J(K) \le Z(J)$ and $J/Z(J) \le \mathrm{SO}(V)$ or $J_0(V)/Z(J_0(V))$ according as $\dim V$ is odd or even. Let $B_K \in \mathrm{Blk}(K)$ and let B_J be a block of J covering B_K. Let D_K and D_J be defect groups for B_K and B_J respectively, chosen with $D_K = K \cap D_J$. Let $Z_c \le Z(K)$ such that $K_c := K/Z_c = \Omega^\eta(V)$, so that $|Z_c| = \gcd(2, q-\eta)$. Then D_K is abelian if and only if D_J is abelian. In addition, if $p = 2$ and D_K is abelian and $J/KZ(J) = J_0(V)/KZ(J_0(V))$, then D_J/D_K is isomorphic to the outer diagonal automorphism group $\mathrm{Outdiag}(K)$ of K.*

Suppose D_K is nonabelian and D_K/Z is SDS for some $Z \leq O_p(Z(K))$. Then $p = 2$ and Z can be taken to be any subgroup of $Z(K)$. If $Z_c \leq Z$, then D_K/Z is given by Tables 2 or 3. If $Z_c \not\leq Z$, then D_K/Z is given by Table 4, where $\delta = \delta_\Gamma = \delta'$ or $2\delta'$ with odd δ' for some $\Gamma \in \mathcal{F}_q^{p'}$ and β is the integer such that $2^\beta \| (q^\delta - \epsilon_\Gamma)$. Thus $\beta = 1$ or a or $a+1$. In the last two cases of the Table 4, $B = B_0(K)$.

Conversely, if $K = Spin(V)$ and K or K/Z is as given in one of Tables 2, 3 and 4 for some $Z \leq O_p(Z(K))$, then there exists a block $B_K \in \mathrm{Blk}(K)$ such that D_K is a defect group for B_K and is as given in Tables 2, 3 and 4.

K	a	D_K	Z	D_K/Z
$\mathrm{Spin}^+(4\delta', q)$	3	$Q_{2^{a+1}} \circ \mathbb{Z}_{2^\beta} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$D_8 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_2$
$\mathrm{Spin}^+(8\delta', q)$	2	$Q_{2^{a+2}} \circ \mathbb{Z}_{2^\beta} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$D_8 \times \mathbb{Z}_{2^{\beta-1}} \times \mathbb{Z}_2$
$\mathrm{Spin}^+(4\delta', q)$	2	$Q_8 \circ \mathbb{Z}_{2^\beta} \times \mathbb{Z}_2$	1	$Q_8 \circ \mathbb{Z}_{2^\beta} \times \mathbb{Z}_2$
$\mathrm{Spin}(4\delta'+1, q)$	2	$Q_8 \circ \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$	1	$Q_8 \circ \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$
$\mathrm{Spin}^+(4, q)$	2	$Q_8 \times Q_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_8$
$\mathrm{Spin}(3, q)$	2	Q_8	1	Q_8

Table 4: Defect 2-groups of $\mathrm{Spin}(V)$ with $Z_c \not\leq Z$ and quotient SDS

Proof: Write $B = B_K$ and $D = D_K$, and $Z_+ \leq Z(D_0(V))$ such that $G = D_0(V)/Z_+ = \mathrm{SO}(V)$, so that $Z_c = Z_+ \cap K$ and $Z_+ \cong \mathbb{Z}_{q-1}$.

Case 1. Suppose p is odd. We may suppose $D \cong DZ_c/Z_c \leq K_c$. Thus D is of defect type in K_c, where a p-subgroup Q of K_c is of defect type if Q is a Sylow p-subgroup of a centralizer $C_{K_c}(t)$ of a semisimple p'-element t. So D is of defect type of G, and D has a primary element $z \in Z(D)$ (see [19, Section 5]). Thus we have the corresponding decompositions $V = V_0 \perp V_+, D = D_0 \times D_+, z = z_0 \times z_+, C_G(z) = L_0 \times L_+$ given by (4.8) and (4.9). In particular, $D_0 = \langle 1_{V_0} \rangle$, $z \in Z(D)$ and $|z| = p$. Let $t = t_0 \times t_+$ with $t_0 \in L_0$ and $t_+ \in L_+$, and let $B_{L_+} = \mathcal{E}_p(L_+, (t_+))$. Then $D_+ \in Syl_p(C_{L_+}(t_+))$ and $D(B_{L_+}) = D_+$.

Suppose first that D is nonabelian and D/Z is SDS for some $Z \leq O_p(Z(K))$. Then $Z = 1$ and $D/Z = D$ is nonabelian, and so D is SDS. But then by Proposition 4.1 (a) and (b), D_+ cannot be SDS, and hence D cannot be SDS after all, a contradiction.

Suppose D is abelian. Since $D = D_J \cap K$ for some D_J and $J/KZ(J)$ is a 2-group, it follows that $D_J \leq KZ(J)$ and $D_J = DO_p(Z(J))$, which is abelian. Conversely, if D_J is abelian, then D is abelian as $D = D_J \cap K$.

Case 2. Suppose $p = 2$, so that q is odd. Suppose that D is nonabelian but D/Z is SDS for some $Z \leq O_p(Z(K))$.

Now B dominates a unique block $B_c \in \mathrm{Blk}(K_c)$ and B_c is covered by a unique block $B_G \in \mathrm{Blk}(G)$, where $G = \mathrm{SO}(V)$. Thus $B_G = \mathcal{E}_2(G, (s))$ for some semisimple $2'$-element s of G^*. If $\dim V = 2n$, then identify G with G^*. If $\dim V = 2n+1$, then identify s with its dual $s^* \in G$. Then a defect group D_G for B_G satisfies $D_G \in \mathrm{Syl}_2(C_G(s))$. Since $[G : K_c] = 2$, it follows that $s \in K_c$ and we may suppose $s \in K$, that is, we identify

$s \in G^* \le J_0(V^*)$ with its dual $s^* \in K \le J_0(V^*)^* = D_0(V)$. This is possible since s is a semisimple $2'$-element.

Write $D_c \cong D/Z_c$, a defect group for B_c. Suppose D_c is abelian. Then D_G is abelian and so $C_G(s^*)$ is a maximal torus of G. By [19, (2E)], $C_{D_0(V)}(s^*)$ is a maximal torus of $D_0(V)$, and so $C_K(s^*) = K \cap C_{D_0(V)}(s^*)$ is a maximal torus of K. But $D \in \mathrm{Syl}_2(C_K(s^*))$, so D is abelian. In the notation of (4.11), $C_G(s^*) = \prod_\Gamma C_\Gamma$, where $C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$ or $I_0(V_{X-1})$. Since $D_G \in \mathrm{Syl}_2(C_G(s^*))$ and D_G is abelian, it follows that $I_0(V_{X-1}) = 1$ or $\mathrm{SO}^\eta(2, q) = \mathrm{GL}^\eta(1, q)$. We may suppose $C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma}) \le I_0(V_\Gamma)$ for all Γ. By [19, (2E)], $C_{D_0(V)}(s^*)$ is a central product of groups L_Γ, where L_Γ is a central extension of C_Γ by Z_+. If $Q_\Gamma = O_2(L_\Gamma)$, then D is a central product of $Q_\Gamma \cap K_\Gamma$, where $K_\Gamma = \mathrm{Spin}(V_\Gamma)$.

Let τ_Γ induce an outer diagonal automorphism of order 2 on $\mathrm{Spin}(V_\Gamma)$. The centralizers $C_{\mathrm{Spin}(V_\Gamma)}(\tau_\Gamma)$ and $C_{\Omega(V_\Gamma)}(\tau_\Gamma)$ are given by [21, Table 4.5.2]. It follows that $C_{\Omega(V_\Gamma)}(\tau_\Gamma) = \mathrm{SL}^{\epsilon_\Gamma}(\delta_\Gamma, q) \circ \frac{1}{2}(q - \epsilon_\Gamma)$ or $\mathrm{SL}^{\epsilon_\Gamma}(\delta_\Gamma, q) \circ (q - \epsilon_\Gamma)$, and $C_{\mathrm{Spin}(V_\Gamma)}(\tau_\Gamma) = \mathrm{SL}^{\epsilon_\Gamma}(\delta_\Gamma, q) \circ (q - \epsilon_\Gamma)$ or $\mathrm{SL}^{\epsilon_\Gamma}(\delta_\Gamma, q) \circ (q - \epsilon_\Gamma) \times 2$ according as δ_Γ is odd or even. In particular,

$$(Q_\Gamma \cap K_\Gamma) \le C_{\mathrm{Spin}(V_\Gamma)}(\tau_\Gamma)$$

and hence D_J is abelian, since $D_J \le \langle DO_p(Z(J)), \tau\rangle$, where $\tau = \prod_\Gamma \tau_\Gamma$. In addition, $\tau \in T_J(B)$ and we may suppose $\tau \in D_J$, and so $D_J/D \cong \mathrm{Outdiag}(K)$.

Suppose D_c is nonabelian. Since $Z(K)$ is a 2-group, it follows that $Z(K) \le Z(D)$. But $D/Z(K) \cong (D/Z)/(Z(K)/Z)$, so $D/Z(K)$ is SDS. Since $D/Z(K) \cong D_c/Z(K_c)$ and since D_c is nonabelian if and only if D_G is nonabelian, it follows by Propositions 4.2 and 4.3 that $\dim V = 2d_\Gamma + 1$ or $4\delta_\Gamma$ according as $\dim V$ is odd or even for some $\Gamma \in \mathcal{F}_q^{p'}$. Moreover, $D/Z(K) = D_c/Z(K_c)$ and $D/Z_c = D_c$ are given in Tables 2 and 3.

Suppose $Z \ne Z_c$ and $Z \ne Z(K)$, so that Z is cyclic of order 1 or 2. Suppose $\Gamma = X - 1$, so that by Tables 2 and 3, $s^* = 1_V$ and B_G is the principal block. If $\dim V$ is odd, then $\Omega_3(q) = PSL_2(q)$, $K = \mathrm{SL}_2(q)$ and so $Z = 1$ and $D/Z = D \cong Q_{2^{a+1}}$ with $a = 2$.

If $\dim V = 4$ and $\eta = -$, then $\Omega_4^-(q) = P\mathrm{SL}_2(q^2)$, $K = \mathrm{SL}_2(q^2)$ and $D \cong Q_{2^{a+2}}$ with $a = 2$. In this case $Z \ne 1$ and so $Z = Z_c = Z(K)$ and D/Z is given by Table 2.

If $\dim V = 4$ and $\eta = +$, then $\Omega_4^+(q) = \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and $K = \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$, $D \cong Q_{2^{a+1}} \times Q_{2^{a+1}}$ and hence $a = 2$. Since D is not SDS, it follows that $Z \ne 1$. Thus $|Z| = 2$ and $Z = Z(\mathrm{SL}_2(q))$, so $D/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_8$.

Suppose $\Gamma \ne X-1$ and so $\delta := \delta_\Gamma = \delta'$ or $2\delta'$ with odd δ', and $C_G(s^*) = \mathrm{GL}^{\epsilon_\Gamma}(2, q^\delta)$. Since $Z(\mathrm{GL}^{\epsilon_\Gamma}(2, q^\delta)) \le K_c$ and $[K{:}K_c] = 2$, it follows that

$$C_{K_c}(s^*) = \mathrm{SL}^{\epsilon_\Gamma}(2, q^\delta) Z(\mathrm{GL}^{\epsilon_\Gamma}(2, q^\delta)).$$

By [19, (2E)], $C_{D_0(V)}(s^*)$ is a central extension of $C_G(s^*)$ by Z_+. If $\dim V = 2n + 1$, then let t be an element of K inducing the central involution of $Z(\mathrm{SL}^{\epsilon_\Gamma}(2, q^\delta))$. By [21, Table 4.5.2], $C_K(t) = \mathrm{Spin}_{2n}^+(q)$ and so

$$C_K(s^*) = C_K(ts^*) = (\mathrm{SL}^{\epsilon_\Gamma}(2, q^\delta) \circ (q^\delta - \epsilon_\Gamma)) \times 2.$$

If $\dim V$ is even, then by [21, Table 4.5.2] again,

$$C_K(s^*) = (\mathrm{SL}^{\epsilon_\Gamma}(2, q^\delta) \circ (q^\delta - \epsilon_\Gamma)) \times 2.$$

It follows that $D \cong Q_{2^{c+1}} \circ \mathbb{Z}_{2^\beta} \times \mathbb{Z}_2$, where $2^{c+1} \| (q^{2\delta} - 1)$ with $c = 2$ or 3.

If $\dim V$ is odd, then $Z = 1$ and hence $a = 2$. If $\dim V$ is even, then $Z = 1$ or $Z = Z(\mathrm{SL}^{\epsilon_\Gamma}(2, q^\delta)) \cong \mathbb{Z}_2$. In the former case $c = a = 2$ and in the later case $c = 3$ and $D/Z = D_8 \times \mathbb{Z}_{\beta-1} \times Z_2$. If $c = 3$, then $a = 3 = c$ and $\delta = \delta'$, or $a = 2$ and $\delta = 2\delta'$. Thus Table 4 holds.

Conversely, suppose $K = \mathrm{Spin}(V)$ and K or K/Z is given in Tables 2, 3 and 4 for some $Z \le Z(K)$. If $\dim V \le 4$, then take $B_K = B_0(K)$. If $\dim V > 4$, then take a $2'$-element $s \in K$ such that $m_\Gamma(sZ_c) = 2$ and let $B_K = \mathcal{E}_2(K, (s))$. Then D_K and D_K/Z are as given in Table 2, 3 or 4. □

We now specialize to the case of extraspecial groups.

Theorem 4.5 *Let K be a finite quasi-simple group of classical type over a field $\mathbb{F}_q$, $B \in \mathrm{Blk}(K)$ and $\eta = \pm$. Let D be a defect group for B. If $p \mid q$, then D/Z is an extraspecial group for some $Z \le O_p(Z(K))$ if and only if $K = K_u/Z_0$ and $D = p_+^{1+2} \in \mathrm{Syl}_p(K)$ and $Z = 1$, where $K_u = \mathrm{SL}^\eta(3, p)$ and Z_0 is any subgroup of $Z(K_u)$.*

Suppose $p \nmid q$. Then D/Z is an extraspecial group for some $Z \le O_p(Z(K))$ if and only if either (K, a, D, Z) is given by Table 5 or $p = 2$, V is orthogonal, $K = \mathrm{Spin}(V)$, $Z_c \le Z(K)$ with $K/Z_c = \Omega(V)$, and $(K/Z_c, a, D/Z_c, Z/Z_c)$ is given by Table 5. Here $\delta = \delta_\Gamma = \delta'$ with $p \nmid \delta'$ or $p = 2$ and $\delta = \delta_\Gamma = 2\delta'$ with odd δ' for some $\Gamma \in \mathcal{F}_q^{p'}$. In addition, if $K = \Omega^+(4\delta' + 1, q)$, then $2 \| (q^{\delta'} - \epsilon_\Gamma)$.

K	a	D	Z
$\mathrm{SL}^\eta(3\delta', q)$	1	3_+^{1+2}	1
$\mathrm{SL}^\eta(2\delta', q)$	2	Q_8	1
$\mathrm{SL}^\eta(2\delta', q)$	3	Q_{2^4}	$O_2(Z(K))$
$\Omega^+(4\delta', q)$	2	Q_8	1
$\Omega^+(4\delta', q)$	2	D_8	1
$\Omega^+(8\delta', q)$	2	Q_{2^4}	$Z(K)$
$\Omega^+(4, q)$	2	2_+^{1+4}	1
$\Omega^+(4\delta' + 1, q)$	2	Q_8	1

Table 5: Extraspecial defect groups of classical groups

Proof: If $p \nmid q$, then the result follows by Propositions 4.1, 4.2, 4.3 and 4.4.

Suppose $p \mid q$, so that D is a Sylow p-subgroup of K. In particular, $|D| = p^{1+2\gamma}$ for some $\gamma \ne 0$. Note that $Z(K_u)$ is a p'-group, so we may take $K = K_u/Z$ for any $Z \le Z(K_u)$.

Now blocks of positive defect of $\mathrm{SL}^\eta(4, q)$ have non-cyclic derived subgroups and so are not SDS. Hence $\mathrm{SL}^\eta(4, q)$ cannot be a subgroup of K. In particular, the Lie rank of K is at most 3. A Sylow p-subgroup of $\mathrm{SL}_2(q)$ is abelian, so $K \not\cong \mathrm{SL}_2(q)$. If $K = \mathrm{SL}^\eta(3, q)$, then D is special with derived subgroup isomorphic to $\mathbb{F}_q$. Thus $q = p$ and $D \cong 3_+^{1+2}$.

If $K = D^\eta_m(q)$, then by [21, Table 2.2], $|D| = q^{m(m-1)}$ and so D cannot be extraspecial. If $K = C_m(q)$ or $B_m(q)$, then $|D| = q^{m^2}$ and so $m = 3$. Since $P\mathrm{Sp}_6(q) \cong \Omega_5(q)$ (with odd q), it follows that we may suppose $K = \mathrm{Sp}_6(q)$ or $\mathrm{SO}_7(q)$. In both cases the derived subgroup of D is non-cyclic, and hence D cannot be extraspecial. □

5 Exceptional groups

We will follow the notation of [21].

Theorem 5.1 *Let K be a finite quasisimple group of exceptional type over a field $\mathbb{F}_q$ and $B \in \mathrm{Blk}(K)$. Let D be a defect group of B. If $p \mid q$, then D is not extraspecial. Suppose $p \nmid q$. Then D/Z is extraspecial for some $Z \leq O_p(Z(K))$ if and only if $p = 3$ and*

$$(K, a, Z, B, D) = (G_2(q), 2, 1, B_0(G_2(q)), 3^{1+2}_+)$$

or $({}^2F_2(2^{2m+1}), 2, 1, B_0({}^2F_2(2^{2m+1})), 3^{1+2}_+)$.

Proof: If $p \mid q$, then $D \in \mathrm{Syl}_p(K)$ and $O_p(Z(K)) = 1$. By [21, Table 2.2], $|D| = q^N$ for some even N or $N = 63$ according as $K \neq E_7(q)$ or $E_7(q)$. In the former case D cannot be extraspecial. In the latter case, K contains a subgroup $\mathrm{SL}^\eta_4(q)$ and so D cannot be extraspecial in this case either.

Suppose $p \nmid q$. For the proof we need to consider not only K but certain overgroups of K. Let $K \lhd H$ such that $C_H(K) \leq Z(H)$, H/K is cyclic and H induces inner-diagonal automorphisms on K. Let B_H be a block of H covering B, and let D_H be a defect group of B_H with $D = K \cap D_H$.

Let K_u be the universal group, so that $K = K_u/Z_0$ for some $Z_0 \leq Z(K_u)$. If $Z(K) \neq \Omega_1(Z(D))$, then take $z \in Z(D) \setminus Z(K)$ with $|z| = p$. If $Z(K) = \Omega_1(Z(D))$, then take $z \in D$ such that $|z| = p^2$ and $zZ(K) \in Z(D/Z(K))$. Let (z, B_z) be a B-subsection, and suppose it is major when $z \in Z(D)$. In any case, let D_z be a defect group for B_z with $D_z \leq D$. Then B_z is a block of $C := C_K(z)$, and D is a defect group for B_z when (z, B_z) is major. By [21, Theorem 4.2.2],

$$C = O^{r'}(C)T, \quad O^{r'}(C) = L_1 \circ L_2 \circ \cdots \circ L_\ell$$

where each $L_i \in \mathcal{L}ie(r)$, and T is an abelian r'-group inducing inner-diagonal automorphisms on each L_i. In general, $z \notin O^{r'}(C)$, so we perform the following modifications. If $Z(C) \leq O^{r'}(C)$, then let $\ell = k$ and $L = O^{r'}(C)$. If $Z(C) \not\leq O^{r'}(C)$, then let $k = \ell + 1$ and $L_k = Z(C)$. Thus

$$C = LT \quad \text{with} \qquad L := L_1 \circ L_2 \circ \cdots \circ L_k, \tag{5.1}$$

$z \in Z(C) \leq L$ and $L \lhd C$. Let B_L be a block of L covered by B_z and $\chi \in \mathrm{Irr}(B_L)$. Note that B_L has defect group $D_L = D_z \cap L$. Then $\chi = \chi_1 \circ \cdots \circ \chi_k$ for some $\chi_i \in \mathrm{Irr}(L_i)$, so that $\chi_i \in \mathrm{Irr}(B_i)$ for some $B_i \in \mathrm{Blk}(L_i)$ and we write $B_L = B_1 \circ B_2 \circ \cdots \circ B_k$.

Since $L = (L_1 \times L_2 \times \cdots \times L_k)/A$ for a central subgroup $A \leq (L_1 \times L_2 \times \cdots \times L_k)$, it follows by [26, Theorems 5.8.8 and 5.8.10] that

$$D_L = (D(B_1) \times D(B_2) \times \cdots \times D(B_k))A/A$$

where $D(B_i)$ is some defect group of B_i and $D(B_i)$ is isomorphic to a subgroup of D_L.

Each element $t \in T$ has the form $t_1 t_2 \cdots t_k t'$, where t' centralizes L and t_i induces an inner-diagonal automorphism on L_i and $[L_i, t_j] = 1$ for $i \neq j$. Let

$$J_i := \langle L_i, t_i : t = t_1 t_2 \cdots t_k t' \in T \rangle$$

and $T' = \langle t' : t = t_1 t_2 \cdots t_k t' \in T \rangle$. Then $LT \lhd J := J_1 \circ J_2 \circ \cdots \circ J_k \circ T'$ and T' is abelian. Let B_J be a block of J covering B_z, so that B_J covers B_L. Thus

$$B_J = B_{J_1} \circ B_{J_2} \circ \cdots \circ B_{J_k} \circ B_{T'}, \tag{5.2}$$

where $B_{J_i} \in \mathrm{Blk}(J_i)$ covering B_i and $B_{T'} \in \mathrm{Blk}(T')$. Let $D(B_{J_i})$ be a defect group of B_{J_i} with $D(B_i) \leq D(B_{J_i})$.

Our strategy is as follows. If D is SDS, then so are D_L and $D(B_i)$ for each i. We treat the exceptional groups case-by-case, progressing from low Lie rank to high (for inductive purposes we must consider the inner-diagonal versions of the groups). In each case we treat the subcases that each $D(B_i)$ is abelian, and that some B_i has nonabelian defect groups. When L_i is classical, this situation has been fully explored in Section 4. When L_i is exceptional, we may use the previously treated exceptional groups of lower Lie rank.

Before treating the exceptional groups case-by-case, we gather together some information.

Suppose $p = 2$, $\ell \geq 2$ and $D(B_i)$ is nonabelian for some i. Then by [21, Table 4.5.2], $\ell = 2$, $Z(C) \leq L_1 \circ L_2$, $k = \ell$, $L = L_1 \circ L_2$ and the possible (K, C) are given in Table 6, where $\eta = -$ or $+$. Here $C = (L_1 \circ L_2).(2{:}2)$ means that $C = \langle L_1 \circ L_2, x \rangle$ such that x induces inner-diagonal automorphism of order 2 on each L_i.

K	C	K	C
$^3D_4(q)$	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q^3)).(2{:}2)$	$G_2(q)$	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).(2{:}2)$
$F_4(q)$	$(\mathrm{SL}_2(q) \circ \mathrm{Sp}_6(q)).(2{:}2)$	$E_6^\eta(q)_u$	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_6^\eta(q)).(2{:}2)$
$E_7(q)_u$	$(\mathrm{SL}_2(q) \circ \mathrm{Spin}_{12}^+(q)).(2{:}2)$	$E_8(q)$	$(\mathrm{SL}_2(q) \circ E_7(q)_u).(2{:}2)$

Table 6: Possible (K, C) with $\ell \geq 2$ and $p = 2$

Suppose $p = 3$ and L_1 is classical. Suppose also that $D(B_1)$ is nonabelian and SDS. By Propositions 4.1, 4.2, 4.3 and 4.4, $L_1 = \mathrm{SL}^{\epsilon_1}(3d_1, q_1)/Z_1$ and $D(B_1) = 3_+^{1+2}$, where $Z_1 \leq Z(\mathrm{SL}^{\epsilon_1}(3d_1, q_1))$, $\gcd(6, d_1) = 1$ and $3 \| (q_1 - \epsilon_1)$. By [21, Table 4.7.3A], $(q_1, \epsilon_1) = (q, \epsilon)$ or $(q^2, 1)$ and (K, C) are given in Table 7, where $L_\epsilon := \mathrm{SL}_3^\epsilon(q)$.

Case 1. *Suppose, moreover that* $K := {}^2B_2(2^{2m+1}), {}^2G_2(3^{2m+1}), {}^2F_4(2^{2m+1}), G_2(q),$ ${}^3D_4(q)$, $F_4(q)$ *or* $E_6^{-\epsilon}(q)$ *with* $q \equiv \epsilon \pmod 3$. *Note that* $Z(K) = 1$, *so* $Z = 1$, *and that* $p \not| [G : H]$, *so* $D_H = D$. *Then* D *is nonabelian and SDS if and only if* $p = 3$ *and*

$$(K, a, Z, B, D(B)) = (G_2(q), 2, 1, B_0(G_2(q)), 3_+^{1+2})$$

or $({}^2F_2(2^{2m+1}), 2, 1, B_0({}^2F_2(2^{2m+1})), 3_+^{1+2})$.

K	C	K	C
${}^3D_4(q)$	$(\mathbb{Z}_{q^2+\epsilon q+1} \circ L_\epsilon).3$	$G_2(q)$	L_ϵ
${}^2F_4(2^{2m+1})$	$\mathrm{SU}_3(2^{2m+1})$	$F_4(q)$	$(L_\epsilon \circ L_\epsilon).(3{:}3)$
$E_6^{-\epsilon}(q)$	$(L_\epsilon \circ \mathrm{SL}_3(q^2)).(3{:}3)$	$E_6^\epsilon(q)_u$	$(L_\epsilon \times L_\epsilon \circ L_\epsilon).(3{:}3{:}3)$
$E_7(q)_u$	$(L_\epsilon \circ \mathrm{SL}_6^\epsilon(q)).(3{:}3)$	$E_8(q)$	$(E_6^\epsilon(q)_u \circ L_\epsilon).(3{:}3)$

Table 7: Possible (K, C) with $D(B_1)$ SDS

Suppose that D is nonabelian and SDS. Since $Z(K) = 1$, it follows that $z \in Z(D)$ (so (z, B_z) is a major subsection) and z induces an inner automorphism on K. Note that each L_i is a classical group (with possibly L_k abelian).

Suppose each $D(B_i)$ is abelian. By Propositions 4.1, 4.2, 4.3 and 4.4, each defect group $D(B_{J_i})$ of each B_{J_i} is abelian and so is a defect group $D(B_J)$ of B_J, since $D(B_J)$ is isomorphic to a quotient group of $D(B_{J_1}) \times \cdots \times D(B_k) \times D(B_{T'})$. But $D_z \cong D(B_J) \cap C$ and (z, B_z) is major, so $D = D_z$ is abelian.

Suppose that $D(B_i)$ is nonabelian for some i. By Propositions 4.1, 4.2, 4.3 and 4.4 again, $p = 2$ or 3. We treat these two cases in turn.

Case 1.1. Suppose $p = 2$. Note that $K \neq {}^2B_2(2^{2m+1})$, ${}^2F_4(2^{2m+1})$, and we may suppose $K \neq {}^2G_2(3^{2m+1})$ since a Sylow 2-subgroup of ${}^2G_2(3^{2m+1})$ is elementary abelian of order 8.

Suppose $\ell \geq 2$. Then $\ell = 2$, $Z(C) \leq L_1 \circ L_2$, $k = \ell$, $L = L_1 \circ L_2$ and the possible (K, C) are given in Table 6.

Let $L_1 := \mathrm{SL}_2(q) \leq J_1 := \langle L_1, x_1 \rangle \leq G_1 = \mathrm{GL}_2^\delta(q)$, $L_2 = \mathrm{SL}_2(q^3), \mathrm{SL}_2(q)$, $\mathrm{Sp}_6(q)$ or $\mathrm{SL}_6^{-\epsilon}(q)$, $J_2 = \langle L_2, x_2 \rangle \leq G_2$ with $G_2 = \mathrm{GL}_2^\delta(q^3)$, $\mathrm{GL}_2^\delta(q)$, $\mathrm{CSp}_6(q)$, $\mathrm{GL}_6^{-\epsilon}(q)$, where δ is the sign such that $2 \| (q - \delta)$ and $x = x_1 \times x_2 \in C \setminus L$ such that x_i induces outer-diagonal automorphism of order 2 on L_i. Then $C \lhd J := J_1 \circ J_2$. If $B_J \in \mathrm{Blk}(J)$ covering B_z, then B_J covering B_L, $B_J = B_{J_1} \circ B_{J_2}$ for some $B_{J_i} \in \mathrm{Blk}(J_i)$ covering B_i, $D(B_J) = D(B_{J_1}) \circ D(B_{J_1})$ and $D_z = D(B_J) \cap C$.

Let B_{G_i} be a block of G_i covering B_{J_i}. Then $B_{G_i} = \mathcal{E}_2(G_i, (s_i))$ for some semisimple $2'$-element $s_i \in G_i^*$. Identify G_i with its dual G_i^* except when $G_i = \mathrm{CSp}_6(q)$, in which case identify s_i with its dual $s_i^* \in G_i$. Thus $D(B_{G_i}) \in \mathrm{Syl}_2(C_{G_i}(s_i))$ and in particular, $|D(B_{G_i}){:}D(B_i)O_2(Z(G_i))| \geq 2$ and hence $|D(B_{J_i}){:}D(B_i)| = 2$ as $D(B_{J_i}) = D(B_{G_i}) \cap J_i$.

Since D is SDS, it follows that each $D(B_i)$ is also SDS. We have treated the case that both $D(B_1)$ and $D(B_2)$ are abelian. We now treat the cases $D(B_1)$ nonabelian and $D(B_2)$ nonabelian.

Suppose that $D(B_1)$ is nonabelian. Since $|D(B_{J_i}){:}D(B_i)| = 2$ for $i = 1$ and 2 and since $D_z = D(B_J) \cap C$, it follows that D_z contains a subgroup which is isomorphic to $D(B_{J_1})$. So $D(B_{J_1})$ is nonabelian and SDS. But by Proposition 4.1 (b), $D(B_{J_1}) \cong SD_{2^4}$, a contradiction.

Suppose that $D(B_2)$ is nonabelian. Then by Proposition 4.3, $K \neq F_4(q)$. A proof similar to above shows that D_z contains a subgroup which is isomorphic to the nonabelian group $D(B_{J_2})$, and by Proposition 4.1, $D(B_{J_2})$ is not SDS, a contradiction. It follows that D cannot be SDS.

Suppose $\ell = 1$. By [21, Table 4.7.1], $L_1 = \mathrm{Spin}_9(q)$ or $\mathrm{Spin}_{10}^{-\epsilon}(q)$ and $C = L_1.2$ or $L_1.(\gcd(4, q+\epsilon))$ according as $K = F_4(q)$ or $E_6^{-\epsilon}(q)$. Writing $Z_1 = \Omega_1(Z(L_1))$, we have $C/Z_1 = \mathrm{SO}_9(q)$ or $\mathrm{SO}_{10}^{-\epsilon}(q).(\gcd(4, q+\epsilon)/2)$. By Proposition 4.4, $D/Z = D = D_z$ is not SDS.

Case 1.2. Suppose $p = 3$ and $D(B_1)$ is nonabelian. By the discussion above $L_1 = \mathrm{SL}^{\epsilon_1}(3d_1, q_1)/Z_1$ and $D(B_1) = 3_+^{1+2}$, where $Z_1 \leq Z(\mathrm{SL}^{\epsilon_1}(3d_1, q_1))$, $\gcd(6, d_1) = 1$ and $3 \| (q_1 - \epsilon_1)$. We have $(q_1, \epsilon_1) = (q, \epsilon)$ or $(q^2, 1)$ and (K, C) are given in Table 7, where $L_\epsilon := \mathrm{SL}_3^\epsilon(q)$.

If $K = G_2(q)$ or ${}^2F_4(2^{2m+1})$, then $\ell = 1$ and $L = C$ and $B_z = B_L = B_0(L)$, so $B = B_0(K)$ with $D(B) = 3_+^{1+2}$.

Let $K = {}^3D_4(q)$, so that $C \cong \mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)} \times H_\epsilon$, where $H_\epsilon = \langle L_\epsilon, x\rangle$ with x inducing outer-diagonal automorphism of order 3 on L_ϵ. Thus $H_\epsilon \leq G_\epsilon = \mathrm{GL}_3^\epsilon(q)$ and $D_z \cong \mathbb{Z}_3 \wr \mathbb{Z}_3 \in \mathrm{Syl}_3(G_\epsilon)$. Since $Z = 1$, it follows that $D/Z = D_z$ is not SDS.

Suppose $K = E_6^{-\epsilon}(q)$ or $F_4(q)$, so that $C = \langle L_1 \circ L_2, x\rangle$, where $L_i = L_\epsilon$ or $\mathrm{SL}_3(q^2)$, and $x = x_1 \times x_2$ such that each x_i induces outer-diagonal automorphism of order 3 on L_i. Let $J_i = \langle L_i, x_i\rangle$, $J = J_1 \circ J_2$ and $B_{J_i} \in \mathrm{Blk}(J_i)$ covering B_i. Let $G_i = \mathrm{GL}_3^\epsilon(q)$ or $\mathrm{GL}_3(q^2)$ such that $J_i \leq G_i$, and let B_{G_i} be a 3-block of G_i covering B_{J_i}. Then $D(B_{J_i}) = D(B_{G_i}) \cap J_i$, and $D(B_{G_i}) \in \mathrm{Syl}_3(C_{G_i}(s_i))$ for some semisimple $3'$-element s_i of G_i. It follows that $D(B_{G_i}) \neq D(B_i)O_3(Z(G_i))$ and so $D(B_{J_i}) \neq D(B_i)$ as $D(B_{J_i}) = D(B_{G_i}) \cap J_i \in \mathrm{Syl}_3(C_{J_i}(s_i))$. If $D(B_i)$ is nonabelian, then $D(B_i) = 3_+^{1+2}$, it follows that $D(B_{J_i}) \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and hence $D_z = (D(B_{J_1}) \circ D(B_{J_2})) \cap C$ contains a subgroup which is isomorphic to $D(B_{J_i})$, which is impossible.

Case 2. *Suppose* $3 \mid (q-\epsilon)$ *and* $K = K_u := 3.E_6^\epsilon(q) \leq H := 3.E_6^\epsilon(q).3$. *Then* D *cannot be nonabelian and SDS. In addition,* D *is abelian if and only if* D_H *is abelian, and* $[D_H{:}D] = 3$ *when* $p = 3$.

Write $m^* := \gcd(m, q-\epsilon)$.

Case 2.1. Suppose $p = 2$. We have $z \in Z(D)$ with $|z| = 2$. By [21, Table 4.5.2], $C := C_K(z) = \langle \mathrm{Spin}_{10}^\epsilon(q) \circ (q-\epsilon), t\rangle$ or given in Table 6, where $t = 4^*{:}1$. By [21, Table 4.5.1], $C_H(z) = \langle \mathrm{Spin}_{10}^\epsilon(q) \circ (q-\epsilon), t_H\rangle$ or $\langle \mathrm{SL}_2(q) \circ \mathrm{SL}_6^\epsilon(q), x_H\rangle$, where $t_H = 4^*{:}3$ and $x_H = 2{:}6$. A proof similar to that of Case 1.1 shows that $D = D_z$ cannot be both nonabelian and SDS. Similarly, since $D = D_z$, it follows by the first part of the proof Case 1 that each $D(B_i)$ is abelian if and only if D is abelian if and only if D_H is abelian (as $D = D_H$).

Case 2.2. Suppose p is odd. Since z is parabolic or equal-rank type and z induces an inner automorphism on K, it follows that each L_i is classical.

Suppose, moreover that each $D(B_i)$ is abelian. Then as in Case 1, D_z is abelian. If $p \geq 5$, then $D_H = D = D_z$ and so both D_H and D are abelian.

Suppose $p = 3$. If $z \in Z(D)$, then (z, B_z) is major and so $D = D_z$ is abelian. Let (z, B_{H_z}) be a B_H-subgroup such that B_{H_z} covers B_z. Then B_{H_z} is a block of $C_H(z)$. By [21, Table 4.7.3A],

$$C_H(z) = \langle \mathrm{SL}_3^\epsilon(q) \times \mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q), 3{:}3{:}1, 1{:}3{:}3\rangle, \quad (\mathrm{SL}_6^\epsilon(q) \circ_{2^*} (q-\epsilon)).(3 \times 2^*),$$

$\mathrm{Spin}_8^+(q) \circ_{2^*} ((q-\epsilon)\times(q-\epsilon)).(2^*\times 2^*\times 3)$, $\mathrm{Spin}_{10}^\epsilon(q)\circ(q-\epsilon).(3\times 2^*\times 2_\epsilon^*)$ (when $q\equiv\epsilon \pmod 9$) with $2_\epsilon^*=1$ or 2^* according as $\epsilon=-$ or $+$, or $(\mathrm{SL}_2(q)\times\mathrm{SL}_5^\epsilon(q))\circ(q-\epsilon).(2^*\times 3)$ (when $q\equiv\epsilon \pmod 9$).

Suppose $C_H(z)=\langle \mathrm{SL}_3^\epsilon(q)\times\mathrm{SL}_3^\epsilon(q)\circ\mathrm{SL}_3^\epsilon(q),t,x\rangle$, so that $L=\mathrm{SL}_3^\epsilon(q)\times\mathrm{SL}_3^\epsilon(q)\circ\mathrm{SL}_3^\epsilon(q)$, $T=\langle t\rangle\le K$ with t induces 3:3:3 on L, and $x\in H\setminus K$ induces 1:3:3 on L. Let $L_i=\mathrm{SL}_3^\epsilon(q)\le G_i:=\mathrm{GL}_3^\epsilon(q)$, $t=t_1t_2t_3$, $x=x_1x_2x_3$ with t_i,x_i act on L_i and centralizes L_j when $i\ne j$. In addition, let $J_i=\langle L_i,t_i,x_i\rangle$, so that $J_i\le G_i$. Let $S\cong\mathbb{Z}_{q-\epsilon}\times\mathbb{Z}_{q-\epsilon}\le\mathrm{SL}_3^\epsilon(q)$ be a maximal torus, and $S\times S\circ_3 S\le L$. Since $C_{\mathrm{GL}_3^\epsilon(q)}(S)=\mathbb{Z}_{q-\epsilon}\times\mathbb{Z}_{q-\epsilon}\times\mathbb{Z}_{q-\epsilon}$ is a maximal torus, it follows that $A:=C_J(S\times S\circ_3 S)$ is abelian such that $A\cap K\simeq\mathbb{Z}_{q-\epsilon}^6$ is a maximal torus of K and $A/(A\cap K)\cong\mathbb{Z}_3$. In particular, we may suppose $t,x\in A$ and $C_H(z)=LA$ with abelian A and $L\lhd C_H(z)$.

Similarly, if $C_H(z)=(\mathrm{SL}_6^\epsilon(q)\times(q-\epsilon)).6^*$, $\mathrm{Spin}_8^+(q)\circ_2((q-\epsilon)\times(q-\epsilon)).(2^*\times 6^*)$, $\mathrm{Spin}_{10}^\epsilon(q)\circ(q-\epsilon).(6^*\times 2_\epsilon^*)$ or $(\mathrm{SL}_2(q)\times\mathrm{SL}_5^\epsilon(q))\circ(q-\epsilon).6^*$, then $A\le C_H(z)$ and so $C_H(z)=LA$ with abelian A and $L\lhd C_H(z)$, and A induces inner-diagonal automorphisms on each L_i.

A proof similar to that in Case 1 with LT replaced by LA and some modifications shows that if each $D(B_i)$ is abelian, then $D(B_{H_z})$ is abelian. Moreover, a proof similar to that of Case 1.2 shows that $|D(B_{H_z}){:}D|=3$. Thus if $z\in Z(D)$, then $D=D_z$ and so $D(B_{H_z})=D_H$, since $|D(B_{H_z}){:}D|=3$ and $H/K\cong\mathbb{Z}_3$. It follows that if each $D(B_i)$ is abelian and $z\in Z(D)$, then D_H is abelian with $|D_H{:}D|=3$. Conversely, if D_H is abelian, then $D=D_H\cap K$ is also abelian.

Case 2.3. Suppose p is odd and $D(B_i)$ is nonabelian for some i, so that as in Case 1, $p=3$ and (K,C) is given in Table 7. A proof similar to Case 1.2 shows that D_z is not SDS. Since $D_z\le D$, it follows that D is not SDS.

Case 2.4. Suppose p is odd and $z\notin Z(D)$, so that $p=3$ and $|z|=9$. If $D(B_i)$ is nonabelian, then by Case 2.3, D is not SDS.

We may suppose each $D(B_i)$ is abelian, $z\in D$ with $|z|=9$ and $zZ(K)\in Z(D/Z(K))$. By [21, Table 4.7.3A], $9\mid(q-\epsilon)$ and $C_H(z)=\mathrm{Spin}_{10}^\epsilon(q)\circ(q-\epsilon).(6^*\times 2_\epsilon^*)$ or $(\mathrm{SL}_2(q)\times\mathrm{SL}_5^\epsilon(q))\circ(q-\epsilon).6^*$. In this case, $C_{H/Z(K)}(zZ(K))$ is also given by [21, Table 4.7.3A], and we have $C_{K/Z(K)}(zZ(K))=C_K(z)/Z(K)$. Thus $D/Z(K)\le C_K(z)/Z(K)$ and $D\le C_K(z)$. In particular, $z\in Z(D)$ and we may suppose (z,B_z) is major. Hence $D=D_z$ is abelian. By Case 2.2, D_H is also abelian.

Case 2.5. Suppose $D=Z(K)\cong\mathbb{Z}_3$, so that $D_H/D=1$ or 3 and D_H is abelian. We claim that $|D_H{:}D|=3$. Let θ be the canonical character of B, so that $D\le\mathrm{Ker}(\chi)$ and so $\theta\in\mathrm{Irr}(K_a)$, where $K_a=K_u/Z(K_u)$. Let K^* be the dual group of K, so that $K^*=K_a.3=\mathrm{Inndiag}(K)$.

Let $\chi\in\mathrm{Irr}(K^*)$ covering θ, and let (s,μ) be the label of χ. Thus s is a semisimple element of K and μ is a unipotent character of $C_K(s)$. So $\chi(1)_3=|K{:}C_K(s)|_3\mu(1)_3$. But $\chi(1)=t\theta(1)$ for some $t\in\{1,3\}$ and $\theta(1)_3=|K_a|_3$, so

$$3\mu(1)_3=t|C_K(s)|_3$$

and $\mu(1)_3=|C_K(s)|_3$ or $|C_K(s)|_3/3$.

If $s=1$, then μ is a unipotent character of K. By [14, pp480, 481], $\mu(1)_3\ne|K|_3$ and $|K|_3/3$. Thus $s\ne 1$. The centralizer $C_K(s)$ and its order are given by [17]. So $O^{r'}(C_K(s))$ is a central product of classical groups of type $A_m^\pm$ or $D_t^\pm$ (with

$t \geq 4$). It follows by the hook-length formula [18, (1.15)] and [27, (22)] that $\mu(1) = 1$, $O^{r'}(C_K(s)) = 1$ and so $C_K(s)$ is a maximal torus of K. In addition, $|C_K(s)|_3 = 3$ and so $O_3(C_K(s)) = Z(K)$. Thus $|K{:}C_K(s)|_3 = |K_a|_3$ and $\theta(1)_3 = \chi(1)_3$. Since $K^*/K_a \cong \mathbb{Z}_3$, it follows that $\chi(1) = \theta(1)$ and hence $\chi|_{K_a} = \theta$. In particular, $T_H(\theta) = T_H(B) = H$. But $|T_H(B){:}D_H| \equiv 1 \pmod 3$ (replacing D_H by a conjugate if necessary), so $|D_H{:}D| = 3$.

Case 3. *Suppose that $K/Z(K) \cong E_7(q)$ with q even. Note that there are no outer diagonal automorphisms of G. If D is SDS, then D is abelian.*

Suppose that D is SDS. Then p is odd, and since $Z(K) = 1$ we have $z \in Z(D)$ (and (z, B_z) is a major subsection). Each L_i is either classical or an exceptional group treated in Case 1 or 2.

Suppose $D(B_i)$ is nonabelian for some i, and recall that $D(B_i)$ is SDS. By Propositions 4.1, 4.2, 4.3 and 4.4 and Cases 1 and 2, $p = 3$, $D(B_i) = 3^{1+2}_+$ and (K, C) is given in Table 7. A proof similar to Case 1.2 shows that $D = D_z$ cannot be SDS after all. It follows that each $D(B_i)$ is abelian. If L_i is classical, then apply Propositions 4.1, 4.2, 4.3 and 4.4. If L_i is exceptional, then apply the results given in Cases 1 and 2. Thus $D = D_z$ is abelian.

Case 4. *Let q be odd, $K = K_u = 2.E_7(q) \leq H := 2.E_7(q).2$. Then D is abelian if and only if D_H is abelian, and moreover, $[D_H{:}D] = 2$ when $p = 2$. If D/Z is nonabelian and SDS, then $p = 2$ and D is given by Table 8 and in addition, $|D_H{:}D| = 2$, where $\beta = 1$ or a. In particular, D/Z is not isomorphic to an extraspecial group for any $Z \leq Z(K)$.*

a	D	Z	D/Z
3	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (Q_{2^{a+1}} \circ \mathbb{Z}_{2^\beta})$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times D_8 \times \mathbb{Z}_{2^{\beta-1}}$
3	$\mathbb{Z}_{2^{a+1}} \circ (Q_{2^{a+1}} \circ \mathbb{Z}_{2^\beta})$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times D_8 \times \mathbb{Z}_{2^{\beta-1}}$
2	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (Q_8 \circ \mathbb{Z}_{2^\beta})$	1	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (Q_8 \circ \mathbb{Z}_{2^\beta})$
2	$\mathbb{Z}_8 \circ (Q_8 \circ \mathbb{Z}_{2^\beta})$	1	$\mathbb{Z}_8 \circ (Q_8 \circ \mathbb{Z}_{2^\beta})$

Table 8: SDS defect 2-groups of $E_7(q)$

As before, write $m^* := \gcd(m, q - \epsilon)$.

Case 4.1. Suppose $p = 2$. Since z induces an inner automorphism on K, it follows by [21, Table 4.5.2] that $C_K(z) \cong \langle \mathrm{SL}_2(q) \circ_{[2s]} \mathrm{Spin}_{12}(q), t\rangle$ with $t = 2{:}s$, $\langle (\mathrm{SL}^\epsilon_8(q)/2) \circ 4, x\rangle$ with $x = (8^*/4){:}1$, or $\langle E^\epsilon_6(q)_u \circ (q-\epsilon), w\rangle$ with $w = 3^*{:}1$, where the sign $\epsilon = \pm$ is chosen so that $4 \mid (q - \epsilon)$. Further, by [21, Tables 4.5.1 and 4.5.2]

$$C_H(z) \cong \langle \mathrm{SL}_2(q) \circ \mathrm{Spin}_{12}(q), t, t_H\rangle, \quad \text{with} \quad t = 2{:}2, \quad t_H = 1{:}2, \tag{5.3}$$

$\langle (\mathrm{SL}^\epsilon_8(q)/2) \circ 4, x_H\rangle$ with $x_H = (8^*/2){:}1$, or $\langle E^\epsilon_6(q)_u \circ (q-\epsilon).2, w\rangle$.

We show that we may take (z, B_z) to be major, and so $D = D_z$. If $z \in Z(D)$, then we have already chosen (z, B_z) to be major. Suppose $z \in D$ such that $|z| = 4$, $z^2 \in Z_0$

and $zZ_0 \in Z(D/Z_0)$, where $Z_0 = Z(K)$. Then $4 \mid (q-\epsilon)$ and

$$C_{H/Z_0}(zZ_0) = \langle (\mathrm{SL}_8^\epsilon(q)/4) \circ 2, \bar{x}_H, \bar{x}\rangle \quad \text{with} \quad \bar{x}_H = (8^*/2){:}1, \quad \bar{x} = \gamma$$

or $\langle 3^*.E_6^\epsilon(q) \circ (q-\epsilon), \bar{w}, \bar{v}\rangle$ with $\bar{w} = 3^*{:}1$ and $\bar{v} = \gamma{:}i$. Here γ and i are graph and inverse automorphisms, respectively. Since $D \le K$ and $D/Z_0 \le C_{H/Z_0}(zZ_0)$, it follows that $D/Z_0 \le C_H(z)/Z_0$, $z \in Z(D)$ and so (z, B_z) is always major, as required.

If each $D(B_i)$ is abelian, then a proof similar to that of Case 1 shows that $D = D_z$ is abelian.

Let (z, B_{H_z}) be a B_H-subgroup such that B_{H_z} covers B_z, so that B_{H_z} covers B_L. Let D_{H_z} be a defect group for B_{H_z} with $D_{H_z} \le D_H$. Suppose each $D(B_i)$ is abelian. A proof similar to that in Case 1 shows that D_{H_z} is abelian. In addition, by Propositions 4.1 and 4.4, $|D_{H_z}{:}D_z| = 2$. Since $D = D_z$ and $[H:K] = 2$, it follows that $D_H = D_{H_z}$, and so D_H is abelian and $|D_H{:}D| = 2$. Since each $D(B_i)$ is abelian when D is abelian, it follows that D is abelian if and only if D_H is abelian.

Suppose D_z is nonabelian and D_z/Z is SDS for some $Z \le Z(K)$.

We have that $\ell \le 2$. Suppose $\ell = 1$, so that $L = L_1 = \mathrm{SL}_8^\epsilon(q)/2$ or $E_6^\epsilon(q)_u$. Thus $D(B_1)$ is nonabelian and $D(B_1)Z/Z$ is SDS. By Proposition 4.1 and Case 2, this is impossible.

Hence $\ell = 2$, so that $L = L_1 \circ L_2$ with $L_1 = \mathrm{SL}_2(q)$ and $L_2 = \mathrm{Spin}_{12}(q)$. If $D(B_2)$ is abelian, then by Proposition 4.4, $|D(B_{J_2}){:}D(B_2)| = 2$. Since $D_z = D$ is nonabelian and D/Z is SDS, it follows that $D(B_1)$ is nonabelian and $D(B_1)Z/Z$ is SDS. So $D(B_1) \in \mathrm{Syl}_2(L_1)$, $B_1 = B_0(L_1)$, $B_{J_1} = B_0(J_1)$ and $D(B_{J_1}) \in \mathrm{Syl}_2(J_1)$. Thus there exists a subgroup $Q \le D_z$ such that $Q \cong D(B_{J_1}) \cong SD_{2^{a+2}}$, which is impossible. Thus $D(B_2)$ is nonabelian.

By Proposition 4.4 and its proof (cf. [17]),

$$D(B_1) \circ D(B_2) \le (\mathrm{SL}_2(q) \circ (\mathrm{SL}_2(q^3) \circ (q^3-\eta))).(2{:}1) \le C_K(z) \tag{5.4}$$

and $D(B_2) \in \mathrm{Syl}_2(\mathrm{SL}_2(q^3) \circ (q^3-\eta))$, where $\eta = \pm 1$. Note that $Z(K) = Z(L)$ is a subgroup of $Z(L_2)$ generated by an element z_s and $L_2/\langle z_s\rangle \not\cong \Omega_{12}(q)$. If $D(B_1)$ is nonabelian, then as shown above D_z contains a subgroup $Q \cong D(B_{J_1}) \cong SD_{2^{a+2}}$, which is impossible. Thus $D(B_1)$ is abelian and hence $D(B_{J_1})$ is abelian. If $G_1 = \mathrm{GL}_2^{\eta'}(q)$ with $2\|(q-\eta')$, then $D(B_{J_1}) \in \mathrm{Syl}_2(C_{G_1}(t))$ for some $2'$-element $t \in G_1$. Thus $D(B_{J_1}) \cong \mathbb{Z}_{2^{a+1}}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $D = D(B_{J_1}) \circ D(B_2)$. Thus D is given by Table 8.

By (5.3) and (5.4),

$$D(B_{H_z}) = D_1 \circ D_2 \le \langle \mathrm{SL}_2(q) \circ (\mathrm{SL}_2(q^3) \circ (q^3-\eta)), 2{:}1, 1{:}2\rangle \le C_H(z), \tag{5.5}$$

where D_1 is an abelian subgroup of $\mathrm{SL}_2(q).2$ and D_2 is a defect group of $(\mathrm{SL}_2(q^3) \circ (q^3-\eta)).2$ containing $D(B_2)$. Thus D_2 is a Sylow 2-subgroup of $(\mathrm{SL}_2(q^3) \circ (q^3-\eta)).2$, $|D_{H_z}{:}D_z| = 2$ and $D_H = D_{H_z}$. In particular we have shown that $[D_H : D] = 2$. Note that $(\mathrm{SL}_2(q^3) \circ (q^3-\eta)).2$ is isomorphic to $\mathrm{GL}_2^\eta(q^3)$ and so $D_2 \cong SD_{2^a+2}$ or $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ according as $2\|(q-\eta)$ or $2^a\|(q-\eta)$.

Case 4.2. Suppose p is odd, so $z \in Z(D)$. Either L_i is classical, or L_i is an exceptional group treated in Case 1 or 2. Suppose each $D(B_i)$ is abelian. If L_i is classical, then apply Propositions 4.1, 4.2, 4.3 and 4.4. If L_i is exceptional, then apply the results given in Cases 1 and 2. Thus $D = D_z$ is abelian. Since $D_H = D = D_z$, it follows that D_H is abelian.

Suppose $D(B_i)$ is nonabelian, so that $D(B_i) \cong 3_+^{1+2}$ and (K, C) is given by Table 7. A proof similar to Case 1.2 with some obvious modifications shows that $D = D_z$ cannot be SDS.

Case 5. *Suppose $K := E_8(q)$, so that $Z(K) = 1$. Then K has no block with extraspecial defect group.*

Since $Z(K) = 1$, we may choose (z, B_z) to be a major subsection of B. If each $D(B_i)$ is abelian, then a proof similar to that in Case 1 shows that $D = D_z$ is abelian. If $D(B_i)$ is nonabelian and p is odd, then $D(B_i) = 3_+^{1+2}$ and (K, C) is given by Table 7. A proof similar to Case 1.2 with some obvious modifications shows that $D = D_z$ cannot be SDS.

Suppose $D(B_i)$ is nonabelian for some i and $p = 2$. Then $C_K(z) \cong \mathrm{Spin}_{16}(q)/\langle z_s \rangle$ or is given by Table 6. In the former case by Proposition 4.4, $D = D_z$ is not isomorphic to an extraspecial group. In the latter case $C_K(z) \cong (\mathrm{SL}_2(q) \circ E_7(q)_u).(2{:}2)$. Thus $L = L_1 \circ L_2$ with $L_1 \cong \mathrm{SL}_2(q)$ and $L_2 \cong E_7(q)_u$. If $D(B_1)$ is abelian, then D contains a subgroup isomorphic to $D(B_{J_1}) \in \mathrm{Syl}_2(L_1.2)$, which is impossible. Thus $D(B_1)$ is abelian and $D(B_{J_1}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_{2^{a+1}}$. In addition, $D(B_2)$ is nonabelian and given by Table 8. Thus $D = (D(B_1) \circ D(B_2)).2$ and as shown in the proof of Case 4.1, D contains a subgroup isomorphic to $SD_{2^{a+2}}$ or $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$, so D is not isomorphic to an extraspecial group. □

To complete the treatment, it remains to consider the simple groups with exceptional covers, namely the prefect groups G where $G/Z(G) \cong A_6$, A_7, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_6(2)$, $Sz(8)$, $\mathrm{Sp}_6(2)$, $\mathrm{O}_7(3)$, $\mathrm{O}_8^+(2)$, $G_2(3)$, $G_2(4)$, $F_4(2)$ and ${}^2E_6(2)$. We take G to be the full cover of $G/Z(G)$.

In none of these cases do we have any new examples of extraspecial groups of $G/O_p(Z)$ (every extraspecial defect group for a faithful block is already a defect group for a non-faithful block). We may use [20] to verify this in all but the cases $F_4(2)$ and ${}^2E_6(2)$ for $p = 3$. For the blocks of the double cover of $F_4(2)$ for $p = 3$, we refer to [22]. For $G/Z(G) \cong {}^2E_6(2)$ and $p = 3$, we use that fact that for every p-subgroup Q we have $C_G(Q)/Z(G) = C_{G/Z(G)}(QZ(G)/Z(G)$, since Q and $Z(G)$ have coprime order, and use an analysis similar to that given in the proof of Theorem 5.1. In each case, every extraspecial defect group for a faithful block is already a defect group for a non-faithful block.

However, we do have new examples of blocks of such groups G with extraspecial defect groups. Using the same references as above: if $G/Z(G) \cong A_6$ or A_7, then $|Z(G)| = 6$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; if $G/Z(G) \cong G_2(4)$, then $|Z(G)| = 2$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; if $G/Z(G) \cong PSL_3(4)$, then $|Z(G)| = 12$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; all other blocks of these groups with extraspecial defect groups are already accounted for.

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