10 Bases

Definition 10.1. *A subset $\mathcal{B}$ of a vector space V is a **basis** for V if $\mathcal{B}$ spans V and $\mathcal{B}$ is linearly independent.*

Examples 10.2. 1. The **standard basis** for K^n is the set $\{e_1, ..., e_n\}$ where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, where the 1 appears in the i-th row. There are many more bases for K^n.

2. Another basis for $\mathbb{R}^2$ is $\{e_1, e_2\}$, where
$$e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3. In $M_{mn}(K)$, the set of matrices E_{ij} with a 1 in the $(i,j)-$th position and 0 everywhere else, for $1 \leq i \leq m, 1 \leq j \leq n$ is the **standard basis** for $M_{mn}(K)$.

4. The set of polynomials $\{1, x, x^2, ..., x^n\}$ form a basis for $\mathcal{P}_n$ and the infinite set $\{1, x, x^2, ...\}$ is a basis for $\mathcal{P}$.

The advantage of a basis over an ordinary spanning set is that every vector in the space has a **unique** representation as a linear combination of the basis vectors:

Theorem 10.3. *Let V be a vector space and let $\mathcal{B}$ be a finite basis for V. Then for every vector $v \in V$ there is a unique expression for v as a linear combination of the vectors in $\mathcal{B}$.*

Proof. Let $\mathcal{B} = \{v_1, ..., v_n\}$. Suppose that
$$v = a_1v_1 +, ... + a_nv_n = b_1v_1 + ... + b_nv_n$$
where $a_1, ..., a_n, b_1, ..., b_n \in K$. Then
$$\underline{0} = v - v = (a_1 - b_1)v_1 + ... + (a_n - b_n)v_n$$
and since $\mathcal{B}$ is linearly independent this gives
$$a_1 - b_1 = ... = a_n - b_n = 0$$
$$\Rightarrow a_i = b_i, \qquad \forall i = 1, ..., n.$$

□

(This proves the theorem for finite $\mathcal{B}$. The proof for an infinite $\mathcal{B}$ is similar, but requires some additional argument.)

Definition 10.4. *A vector space V is called **finite dimensional** if it has a finite basis.*

Lemma 10.5. *Let V be a finite dimensional vector space. Let S be a finite spanning set for V and S_0 be a linearly independent subset of S. Then S contains a basis $\mathcal{B}$ of V with $S_0 \subseteq \mathcal{B}$.*

Proof. Let $\mathcal{B}$ be a maximum linearly independent subset of S that contains S_0, ie. $\mathcal{B}$ is not contained in any larger linearly independent subset of S that contains S_0. We need to show that $\mathcal{B}$ is a spanning set for V. If $\mathcal{B} = S$, then we are done, so assume that $\mathcal{B} \neq S$. Label the elements of S so that $\mathcal{B} = \{v_1, ..., v_n\}, S = \{v_1, ..., v_m\}$ where $m > n$. The idea of the remainder of the proof is that (by maximality of $calB$) each element of S may be written as a linear combination of elements of $\mathcal{B}$, and each element of V may be written in terms of the elements of S, so every element of V may be written in terms of elements of $\mathcal{B}$. To do this thoroughly requires a bit of notation.

Let $j \in \{n+1, \ldots, m\}$. By maximality of $\mathcal{B}$ as a linearly independent set $\mathcal{B} \cup \{v_j\}$ is a linearly dependent set. Hence there is a non-trivial combination

$$a_{j,1}1v_1 + ... + a_{j,n}v_n + a_j v_j = \underline{0}$$

where $a_j \neq 0$ because $\mathcal{B}$ is linearly independent. Therefore

$$v_j = -a_j^{-1}(a_{j,1}v_1 + ... + a_{j,n}v_n).$$

We have shown that v_j is a linear combination of the vectors in $\mathcal{B}$ for all $j = n+1, ..., m$. Now consider any $v \in V$. Since S is a spanning set we can write

$$\begin{aligned} v &= \sum_{i=1}^{m} b_i v_i \\ &= \sum_{i=1}^{n} b_i v_i + \sum_{j=n+1}^{m} b_j v_j \\ &= \sum_{i=1}^{n} b_i v_i + \sum_{j=n+1}^{m} b_j \sum_{i=1}^{n} (-a_j^{-1} a_{j,i}) v_i \\ &= \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} \left(\sum_{j=n+1}^{m} -b_j a_j^{-1} a_{j,i} \right) v_i \end{aligned}$$

$$= \sum_{i=1}^{n} \left(b_i - \sum_{j=n+1}^{m} b_j a_j^{-1} a_{j,i} \right) v_i.$$

Therefore v can be written as a linear combination of vectors in $\mathcal{B}$ and $\mathcal{B}$ is a spanning set for V. □

Theorem 10.6. *Every finite linearly independent set in a finite dimensional vector space V can be extended to form a basis for V.*

Proof. Let S_0 be a finite linearly independent set in V and let T be any finite spanning set for V. Then $S = S_0 \cup T$ is a finite spanning set for V containing S_0. By Lemma 10.5, S contains a basis $\mathcal{B}$ of V with $S_0 \subseteq \mathcal{B}$. □

The next result is required to prove the main theorem of this section:

Lemma 10.7. *Let V be a finite dimensional vector space. Let $R = \{u_1, ..., u_n\}$ be a linearly independent set in V and $S = \{v_1, ..., v_m\}$ be a spanning set for V. Then $n \leq m$.*

Proof. Consider the set $T_1 = \{u_1\} \cup S$. Then T_1 is a spanning set because it contains S and it is linearly dependent because u_1 is a linear combination of elements of S. Therefore by Lemma 10.5 T_1 contains a basis $\mathcal{B}_1$ containing the linearly independent set $\{u_1\}$. Now $\mathcal{B}_1$ is a proper subset of T_1 because T_1 is linearly dependent, and so $\mathcal{B}_1 = \{u_1\} \cup S_1$ for some proper subset S_1 of S.

Now consider $T_2 = \{u_2\} \cup \mathcal{B}_1 = \{u_1, u_2\} \cup S_1$. Then T_2 is a spanning set and it is linearly dependent because $u_2 \in \text{span}(\mathcal{B}_1)$. Therefore T_2 contains a basis $\mathcal{B}_2$ containing the linearly independent set $\{u_1, u_2\}$ and $\mathcal{B}_2$ is a proper subset of T_2. We can write $\mathcal{B}_2 = \{u_1, u_2\} \cup S_2$ where S_2 is a proper subset of S_1.

Continuing in this way we find that V has a basis $\mathcal{B}_n = \{u_1, ..., u_n\} \cup S_n$ where

$$S_n \subsetneq S_{n-1} \subsetneq ... \subsetneq S_1 \subsetneq S.$$

Hence

$$0 \leq |S_n| \leq |S| - n$$
$$\Rightarrow n \leq |S| = m.$$

□

Theorem 10.8. ***(The Basis Theorem)*** *Any two bases of a finite dimensional vector space have the same number of elements.*

Proof. Let V be a finite dimensional vector space . Then V has a finite basis $\mathcal{B}$ by definition. Suppose $\mathcal{B}$ has n elements. Let $\mathcal{C}$ be any other finite basis and suppose that $\mathcal{C}$ has m elements. Since $\mathcal{B}$ is linearly independent and $\mathcal{C}$ is a spanning set, then Lemma 10.7 gives $n \leq m$. Similarly $m \leq n$ since $\mathcal{B}$ is a spanning set and $\mathcal{C}$ is linearly independent. Therefore $n = m$.

Finally, any basis of V is finite, otherwise V would contain an infinite linearly independent set, and hence a finite linearly independent set with more than n elements. However this contradicts Lemma 10.7 because $\mathcal{B}$ is a spanning set with n elements. □

Definition 10.9. *Let V be a finite dimensional vector space over K, with $V \neq \{\underline{0}\}$. The number of vectors in any basis for V is called the* ***dimension*** *of V, denoted by $dim_K(V)$ or simply $dim(V)$. For the zero space we set $dim\{\underline{0}\} = 0$.*

Examples 10.10.

1. For any $n \geq 1$ and any field K,
$$dim_K(K^n) = n.$$

2. The set $\{1, x, ..., x^n\}$ is a basis for $\mathcal{P}_n$. Hence $\dim(\mathcal{P}_n) = n + 1$.

3. We have seen that the set of matrices $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ forms a basis for the space of symmetric matrices in $M_2(K)$. Hence this space has dimension 3. It is a subspace of $M_2(K)$.

4. The matrix space $M_{mn}(K)$ has standard basis
$$\{E_{ij} | \, 1 \leq i \leq m, 1 \leq j \leq n\}$$
where E_{ij} is the matrix with (i, j)-th entry 1 and all other entries 0. Therefore
$$dim_K(M_{mn}(K)) = mn.$$

5. Indicating the field K can be important because some sets can be regarded as vector spaces over different fields. For example if $V = \mathbb{C}$, then
$$dim_{\mathbb{C}}(\mathbb{C}) = 1, \text{ with basis } \{1\},$$
$$dim_{\mathbb{R}}(\mathbb{C}) = 2, \text{ with basis } \{1, i\}.$$

Summary: Let V be a finite dimensional vector space over a field K. Then

1. V has a finite basis and all bases of V have the same (finite) number of elements. This number is called the dimension of V.
2. Every finite spanning set for V contains a basis.
3. Every linearly independent set in V can be extended to form a basis.

The following theorem is an easy consequence of these important facts.

Theorem 10.11. *Let V be a vector space with $dim(V) = n$. Then*

(i) any linearly independent set in V contains at most n vectors;

(ii) any spanning set for V contains at least n vectors;

(iii) any linearly independent set with exactly n vectors is a basis;

(iv) any spanning set of exactly n vectors is a basis.

Proof. (i) If S is a linearly independent subset of V, then S is contained in a basis $\mathcal{B}$ and so $|S| \leq |\mathcal{B}| = n$.

(ii) If S is a spanning set for V, then S contains a basis $\mathcal{B}$ and $n = |\mathcal{B}| \leq |S|$.

(iii) If S is a linearly independent set with $|S| = n$, then (i) implies that $S = \mathcal{B}$ for some basis $\mathcal{B}$.

(iv) If S is a spanning set for V with $|S| = n$, then (ii) implies that $S = \mathcal{B}$ for some basis $\mathcal{B}$.

□

Remark: Let V be a vector space of dimension n. Then if we are given a set of n vectors $\mathcal{B}$, to prove $\mathcal{B}$ is a basis we only need to show ONE of the following: either $\mathcal{B}$ is a spanning set OR $\mathcal{B}$ is linearly independent.

The next result shows that any subspace of a finite dimensional vector space is also finite dimensional.

Theorem 10.12. *Let W be a subspace of a finite dimensional vector space V. Then*

(i) W is finite dimensional and $dim(W) \leq dim(V)$;

(ii) $dim(W) = dim(V)$ if and only if $W = V$.

Proof. (i) Let $dim(V) = n$. Since $W \subseteq V$, it cannot contain more than n linearly independent vectors by Theorem 10.11(i) Let $\mathcal{C} = \{w_1, ..., w_m\}$ be a basis for W. Then $\mathcal{C}$ is linearly independent in V and therefore

$$dim(W) = m \leq n = dim(V).$$

(ii) If $dim(W) = dim(V) = n$, then W has a basis with n vectors and this is also a basis for V. Therefore $W = V$, Conversely

$$W = V \Rightarrow dimW = dimV.$$

□

Theorem 10.13. *Let W_1 and W_2 be finite dimensional subspaces in a vector space V. Then*

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

Proof. Since $W_1 \cap W_2$ is a subspace of W_1 and of W_2 it is finite dimensional by Theorem 10.12(i). Let $\mathcal{B} = \{u_1, ..., u_n\}$ be a basis for $W_1 \cap W_2$. Then this can be extended to a basis $\mathcal{B}_1 = \{u_1, ..., u_n, v_1, ..., v_k\}$ for W_1 and a basis $\mathcal{B}_2 = \{u_1, ..., u_n, w_1, ..., w_m\}$ for W_2. We will show that

$$\mathcal{B}_3 = \mathcal{B}_1 \cup \mathcal{B}_2 = \{u_1, ..., u_n, v_1, ..., v_k, w_1, ..., w_m\}$$

is a basis for $W_1 + W_2$. It is clear that $\mathcal{B}_3$ spans $W_1 + W_2$ so it remains to show that it is linearly independent. Assume that

$$a_1u_1 + ... + a_nu_n + b_1v_1 + ... + b_kv_k + c_1w_1 + ... + c_mw_m = \underline{0} \qquad (*)$$

for some scalars $a_1, ..., a_n, b_1, ..., b_k, c_1, ..., c_m \in K$. Then $u + v + w = \underline{0}$, where $u = a_1u_1 + ... + a_nu_n, v = b_1v_1 + ... + b_kv_k, w = c_1w_1 + ... + c_mw_m$. So $w = -u - v \in W_1 \cap W_2$ and we can write $w = d_1u_1 + ... + d_nu_n$ for some $d_1, ..., d_n \in K$. Substituting into $(*)$ we get

$$(a_1 + d_1)u_1 + ... + (a_n + d_n)u_n + b_1v_1 + ... + b_kv_k = \underline{0}. \qquad (**)$$

Since $\mathcal{B}_1$ is linearly independent, all coefficients in $(**)$ are zero and so $(*)$ becomes

$$a_1u_1 + ... + a_nu_n + c_1w_1 + ... + c_mw_m = \underline{0}. \qquad (***)$$

Since $\mathcal{B}_2$ is linearly independent, all coefficients in $(***)$ are zero and this means that all coefficients in $(*)$ are zero. Therefore $\mathcal{B}_3$ is linearly independent and is a basis for $W_1 + W_2$.

Finally we have

$$dim(W_1 + W_2) = n + k + m = (n + k) + (n + m) - n$$

$$= dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

□

Corollary 10.14. *If W_1, W_2 are finite dimensional subspaces of a vector space V such that $W_1 \cap W_2 = \{\underline{0}\}$, then $dim(W_1 + W_2) = dim(W_1) + dim(W_2)$.*

Example 10.15. In $M_2(\mathbb{R})$ let

$$W_1 = \{A \in M_2(\mathbb{R}) \mid A = A^T\},$$

$$W_2 = \{A \in M_2(\mathbb{R}) \mid A = -A^T\}$$

Then $dim(W_1) = 3$ (see Examples 10.10) and $dim(W_2) = 1$ (with basis $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$).

Also $W_1 \cap W_2 = \{\underline{0}\}$ and so $dim(W_1 + W_2) = 4 = dim(M_2(\mathbb{R}))$.
Therefore $W_1 + W_2 = M_2(\mathbb{R})$.