MATH10111 - SETS, NUMBERS AND FUNCTIONS

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Every week (except Week 6, which is Reading Week) you should attend a 50 minute **Supervision Group** associated with this course. The aim of these supervisions (which typically have at most 10 students) is to sort out any difficulties with the lecture material and to get feedback on the answers to the **starred questions** which you have handed in.

Before your supervision in Week 1, please attempt questions 1.1, 1.2, 1.3 and take your solutions to this first supervision. You will not necessarily have had any lectures yet, but having attempted these questions you will be prepared to discuss them in your supervision groups.

For subsequent weeks, your supervisor will tell you which starred questions you are to attempt and when/where to hand them in so that they can be marked before the next supervision.

You should aim to do **ALL** the questions set for this course. Frequently, the un-starred questions are "practice runs" for the starred questions. For example, Questions 2.1 and 2.2 are of a similar nature to Questions 2.3 and 2.4 (which are starred questions). So, I suggest you first try Questions 2.1 and 2.2 and if you are really stuck you can find solutions to these questions in the course text (called [IMR]). Now you can go on to produce beautiful solutions to Questions 2.3 and 2.4.

Attached are the questions for Weeks 1 to 5 (or W1-W5 for short) and information for the first half of the course, including a course test.

MATH10111 - SETS, NUMBERS AND FUNCTIONS

COURSE TEXT:

"An Introduction to Mathematical Reasoning" by Peter Eccles (Cambridge University Press). [IMR]

LECTURE NOTES	[IMR]	WHEN?
1 INTRODUCTION	-	-
2 MATHEMATICAL LOGIC	CHAPTERS 1,2	W1
3 PROVING THINGS (PROOF BY CONTRADICTION)	CHAPTER 4	W2
4 PROOF BY INDUCTION	CHAPTER 5	W3
5 SET THEORY	CHAPTERS 6,7	W4
6 FUNCTIONS	CHAPTERS 8,9	W5

W6: READING WEEK (NO LECTURES)

IN W7 THERE WILL BE A COURSE TEST

W7-W12: INFORMATION TO FOLLOW

MATH10111 - EXERCISES

WEEK 1

(QUESTIONS MARKED * TO BE HANDED IN TO SUPERVISORS)

1.1 BUY "AN INTRODUCTION TO MATHEMATICAL REASONING"!

***1.2** A *proposition* is a sentence which is about specific mathematical objects and which is either true or false (but not both). A *predicate* is a statement which gives a property that an object may or may not have.

Which of the following are propositions and which are predicates? In each case write down the negation of the statement.

(i) $2^{10} - 1$ is a prime number.
(ii) The largest angle of triangle x is less than or equal to $\pi/2$.
(iii) The real number x is such that $x^2 < x$.
(iv) All natural numbers divisible by 2 are divisible by 4.

***1.3** Work out the truth table for each of the following ($\neg$ means the negation of a statement):

(i) $(p \vee q) \vee (\neg p)$;
(ii) $p \wedge (\neg p)$;
(iii) $\neg[(p \wedge q) \vee (\neg p \ \wedge \ \neg q)]$;
(iv) $\neg[(p \vee q) \wedge (\neg p \ \vee \ \neg q)]$;
(v) $p \Rightarrow (p \vee q)$;
(vi) $\neg p \Leftrightarrow (p \Rightarrow \neg q)$.

Are (iii) and (iv) equivalent?

1.4 Read Chapter 3 of [IMR] and do the following questions: 3.1(i), 3.2, 3.3, 3.4 (all on page 29 of [IMR]).

MATH10111 - EXERCISES

WEEK 2

2.1 Question 4.1 of [IMR]

2.2 Question 4.2 of [IMR]

***2.3** Prove by contradiction that there do not exist integers m and n such that

$$24n + 3m^2 = 7.$$

***2.4** Prove that for every non-zero real number x,

$$(x + x^2)^2 \neq (x - x^2)^2.$$

***2.5** Let n be a natural number which is not a square (that is, there is no natural number m such that $n = m^2$). Prove that $\sqrt{n}$ is not a rational number.

[Hint: Take inspiration from proofs in the lecture notes.]

MATH10111 - EXERCISES

WEEK 3

3.1 Question 5.1 of [IMR]

3.2 Question 5.2 of [IMR]

***3.3** Prove by induction on n that 13 divides $2^{4n+2} + 3^{n+2}$ for all natural numbers n.

***3.4** Prove by induction on n that $n! > 2^n$ for all natural numbers n such that $n \geq 4$.

***3.5** Prove by induction on n that for all natural numbers n

$$\sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2.$$

***3.6** Let (u_n) be the sequence of numbers defined by

$u_1 = 1$

$u_2 = 1,$

$u_{k+1} = u_{k-1} + u_k$ for $k \geq 2$

(These are the Fibonacci numbers - see page 49 of [IMR]).

Prove by induction on n that

$$u_n^2 = u_{n-1}u_{n+1} + (-1)^{n-1}$$

for all natural numbers n such that $n \geq 2$.

MATH10111 - EXERCISES

WEEK 4

***4.1** Mark the following statements as true or false:
(i) $\frac{7}{2} \in \mathbb{Z}$;
(ii) $\pi \subseteq \mathbb{R}$;
(iii) $\{1, -1\} \subseteq \mathbb{R}$;
(iv) $\{\{1\}\} = \{1\}$;
(v) $\{3, 2, 2, 6, 9, 8\} \subseteq \{1, 2, 6, 7, 8, 3, 9, 10, 10\}$;
(vi) $\{0, 17, 31\} \not\subseteq \mathbb{N}$;
(vii) $\{\pi\} \notin \mathbb{Z}$.

***4.2** Let A, B, D and E be the following sets: $A = \{1, 2, 3, 4\}$; $B = \{\pi, \sqrt{2}, e\}$; $D = \{2k : k \in \mathbb{Z}\}$; $E = \{1, 3, 5\}$.
Work out the following sets:
(i) $A \cap B$;
(ii) $A \cup D$;
(iii) $A \cup B$;
(iv) $\mathcal{P}(A)$;
(v) D^c where the universal set is $\mathbb{Z}$;
(vi) E^c where the universal set is $\{x : x \in \mathbb{N}, x \leq 12\}$;
(vii) $A \setminus E$;
(viii) $D \setminus A$;
(ix) $A \times E$.

4.3 Question 6.4 of [IMR].

***4.4** By using a truth table prove that for sets A, B and C

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

4.5 Question 6.2 of [IMR].

***4.6** For sets A and B, prove that $(A \cup B)^c = A^c \cap B^c$.

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WEEK 5

5.1 Question 7.2 of [IMR].

***5.2** Question 11, page 117 of [IMR].

***5.3** Which of the following are functions? In each case, if f is not a function then modify the range or domain to produce a function.
(i) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \frac{1}{1+x^2}$.
(ii) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \frac{1}{1-x^2}$.
(iii) $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = \frac{1}{2}x$.
(iv) $f : [-1, 1] \to \mathbb{R}$ with $f(x) = \sin^{-1} x$.
(v) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \log_e(x)$.

5.4 Question 9.1 of [IMR].

***5.5** (a) Which of the following functions are (1-1)?
(i) $f : \mathbb{N} \to \mathbb{N}$ with $f(x) = x^3$.
(ii) $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = e^{-x}$.
(iii) $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^4$.

(b) Which of the following functions are onto?
(i) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$.
(ii) $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $g(x) = \frac{1}{x^3}$.
(iii) $h : \mathbb{Z} \to \mathbb{Z}$ with $h(x) = x - 3$.

***5.6** Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 - 2|x|$ and let $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = 3x-2$. Find formulas for (i) $f \circ f$ (ii) $g \circ g$ (iii) $f \circ g$ (iv) $g \circ f$
In each case determine the image of -1 under the composite function.

***5.7** Let $f : A \to B$ and $g : B \to C$ be functions. Prove that
(i) if $g \circ f$ is (1-1), then f is (1-1);
(ii) if $g \circ f$ is onto, then g is onto.

Give an example where $g \circ f$ is both (1-1) and onto but g is not (1-1) and f is not onto.